Math 31: Topics in Algebra
Summer 2019 - Problem Set 3

Due: Wednesday, July 17

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#**1** (Normal Subgroups)

a) Recall that given a group G, the *center* of G, denoted $Z(G)$, is the set of elements that commute with all other elements in G. We've seen previously that $Z(G) \leq G$. Prove $Z(G) \trianglelefteq G$.

b) Let G be a group. Suppose H and K are subgroups of G and that K is normal in G. Prove that HK is a subgroup of G, where HK is the set of elements of G that can be written as a product hk for $h \in H$ and $k \in K$.

#**2** Find a subgroup $G \leq S_6$ such that $G \cong A_3 \times \mathbb{Z}_2$.

#**3** (Partitions and Equivalence Relations)

a) Let G be a group. Prove that the following defines an equivalence relation: $x \sim y$ if and only if there exists a positive integer k such that $x^k = y^k$.

b) Let G be a group. Prove that the following defines an equivalence relation: $x \sim y$ if and only if $\mathrm{ord}(x) = \mathrm{ord}(y)$.

c) Write down the equivalence classes of S_4 given by the equivalance relation in b).

#**4** (Cosets and Lagrange's Theorem)

a) Prove that if G is a group of order n, then $g^n = e$ for all $g \in G$.

b) Let H and K be subgroups of a group G, with $|H| = n$, $|K| = m$, $\gcd(n, m) = 1$. Prove that $H \cap K = \{e\}$.

c) Let G be a group, $g \in G$, and $H \leq G$. Prove that $Hg = H$ if and only if $g \in H$.
