Math 31: Topics in Algebra
Summer 2019 - Problem Set 4

Due: Wednesday, July 24

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#**1** Find a subgroup of $G \leq S_5$ such that $G \cong D_5$. You don't have to prove your function is a homomorphism, but do write each of the elements in G in disjoint cycle notation.

(Hint: Cayley's permutation representation won't work here. Instead, label the vertices of the regular pentagon 1, 2, 3, 4, 5 similarly to what we did with the square in class. To which permutations should r and s go? These two elements should satisfy the same equations in S_5 (and thus also in G) that r and s satisfy in D_5.)

#**2** *(Second Isomorphism Theorem)* For this problem, let G be a group, with $H \trianglelefteq G$ and $K \leq G$. We know from PS 3 that $HK \leq G$ (actually, you proved it when K is normal and H isn't, but it works the same way.)

a) Prove that $H \cap K \trianglelefteq K$ and $H \trianglelefteq HK$.

b) By definition, the quotient group HK/H consists of cosets $\{Hg : g \in HK\}$. Prove (observe, really) that for every $g \in HK$, there exists some $k \in K$ such that $Hg = Hk$. Thus every coset of this quotient group may be represented as Hk for some $k \in K$.

c) Define $f : K \to HK/H$ by $f(k) = Hk$. Prove that this is a surjective homomorphism with kernel $H \cap K$ (what is the identity element in HK/K?). Conclude that $HK/H \cong K/(H \cap K)$. Interpret this result in the case that $H \cap K = \{e\}$.

(Aside: Pinter refers to this as the First Isomorphism Theorem, but it's generally known as the Second Isomorphism Theorem. The First Isomorphism Theorem usually refers to what Pinter calls the Fundamental Homomorphism Theorem, and the Third Isomorphism Theorem usually refers to what Pinter calls the Second Isomorphism Theorem.)

#**3** (Proving groups are isomorphic using quotients) For the following problems, do NOT attempt to write down an explicit isomorphism. Instead, find a surjective homomorphism and use the FHT. You do not need to prove that any of the subgroups in question are normal (but do show that your constructed functions are surjective homomorphisms with the desired kernel)..

a) Recall that $\mathrm{SL}_n(\mathbb{R})$ is the group of n-by-n matrices with real entries and determinant 1, a subgroup of $\mathrm{GL}_n(\mathbb{R})$. Prove that $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(R) \cong \mathbb{R}^*$ for all positive integers n.

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b) Prove that $S_n/A_n \cong \mathbb{Z}_2$ for $n \geq 2$. (It's easy to observe this by arguing that $(S_n : A_n) = 2$, but do actually construct a homomorphism!)

#**4** (Abelian Groups)

a) Prove that if G is an abelian group and $H \leq G$, then G/H is an abelian group.

b) List all abelian groups of order 360, up to isomorphism, as a product of cyclic groups.

c) Let m and n be positive integers. Show that there is a surjective homomorphism $f : \mathbb{Z}_n \to \mathbb{Z}_m$ if and only if n is a multiple of m.