Math 31: Topics in Algebra
Summer 2019 - Problem Set 5

Due: Wednesday, July 31

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#**1** Let A be any nonempty set and let $\mathcal{P}(A)$ be its power set.

a) Prove that $\mathcal{P}(A)$ is a ring, with addition defined by symmetric difference and multiplication defined by intersection. Is it commutative ring?

b) Find all zero divisors of $\mathcal{P}(A)$. Is it an integral domain?

c) Find all units of $\mathcal{P}(A)$. Is it a field?

#**2** (Properies of Rings)

a) Prove that the only ideals of a field F are F and $\{0\}$.

b) Prove that every field is an integral domain.

c) Let A be a ring. An element $x \in A$ is *idempotent* if $x^2 = x$. Prove that if every element is idempotent, then A is commutative.

#**3** (Ideals and Subrings)

a) Let A be a commutative ring with 1 and J an ideal of A. Define $\text{Rad}(J) = \{x \in A : x^n \in J \text{ for some } n \in \mathbb{Z}^+\}$. Prove $\text{Rad}(J)$ is an ideal of A.

b) Let A be a ring and $f : A \to A$ a ring homomorphism. Define $\text{Fix}(f) = \{x \in A : f(x) = x\}$. Prove $\text{Fix}(f)$ is a subring of A.

c) Let A be a ring and $f : A \to B$ a homomorphism. Prove that if A is an integral domain, then $f(1_A) = 1_B$ or $f(1_A) = 0_B$. Prove also that if $f(1) = 1$, then the image of every unit in A is a unit in B, and if $f(1) = 0$, then $f(x) = 0$ for all $x \in A$.

#**4** (Quotient Rings) Let A be a commutative ring with unity, and let J be an ideal of A.

a) Prove that A/J has unity if and only if there exists an element $a \in A$ such that $ax - x \in J$ and $xa - x \in J$ for every $x \in A$.

b) Prove that J is a prime ideal if and only if A/J is an integral domain.
