Math 31: Topics in Algebra
Summer 2019 - Problem Set 6

Due: Wednesday, August 7

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#**1** Let A be a *principal ideal domain*: an integral domain in which every ideal is principal (eg. we proved that $\mathbb{Z}$ is a PID using the division algorithm). Let $J = \langle x \rangle$ be a prime ideal of A. Show that J is maximal (if $J \subseteq I$ for some ideal I, then $I \in \{J, A\}$).

#**2**

a) Let A be a finite commutative ring with unity. Prove that every nonzero element is either a unit or a zero divisor.

b) Let A be a finite integral domain. Show that if there exist some nonzero $x, y \in A$ such that $(x+y)^2 = x^2 + y^2$, then $\text{char}(A) = 2$.

#**3**

a) Prove that for all nonzero integers a and b, $\gcd(a, b) = \gcd(a, b + ax)$ for all $x \in \mathbb{Z}$.

b) Recall that for two ideals I and J in a ring A, $I + J = \{x + y : x \in I, y \in J\}$ is also an ideal of A. Prove that for nonzero integers a and b wth $\gcd(a, b) = d$, $\langle a \rangle + \langle b \rangle = \langle d \rangle$.

#**4**

a) Prove that if $a^2 \equiv 1 \pmod 2$ then $a^2 \equiv 1 \pmod 4$.

b) Let p and q be distinct primes. Prove $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.
