Math 31: Topics in Algebra
Summer 2019 - Problem Set 7

Due: Wednesday, August 14

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#**1** How many polynomials of degree exactly n are there over $\mathbb{Z}_p$, where p is prime and n is a non-negative integer?

#**2** (Polynomial Division)

a) Find the quotient $q(X)$ and remainder $r(X)$ when $a(X) = X^3 + X^2 + X + 1$ is divided by $b(X) = X^2 + 3X + 2$ in $\mathbb{Q}[X]$ and $\mathbb{Z}_5[X]$. Verify your answers by confirming that $a(X) = b(X)q(X) + r(X)$.

b) Find the quotient $q(X)$ and remainder $r(X)$ when $a(X) = X^3 + 2$ is divided by $b(X) = 2X^2 + 3X + 4$ in $\mathbb{Q}[X]$, in $\mathbb{Z}_3[X]$, and $\mathbb{Z}_5[X]$. Verify your answers by confirming that $a(X) = b(X)q(X) + r(X)$.

#**3** (Polynomials over a Ring with Zero Divisors)

a) Find all ways to factor X^2 into a product of two degree one polynomials over $\mathbb{Z}_9$ and over $\mathbb{Z}_7$. Why are these answers different?

b) Show that in $\mathbb{Z}_8[X]$, there are infinitely many square roots of 1.

#**4** (Polynomials over a Field)
Let F be a field and let J be an ideal of $F[X]$. Recall that $F[X]$ is a principal ideal domain: all ideals are principal.

a) Prove that any two generators of J are associates, and so J has a unique monic generator.

b) Prove that J is a prime ideal if and only if it has an irreducible generator.

#**5**

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a) Factor the following polynomials into irreducibles over $\mathbb{Z}_5$:

- $X^3 + X^2 + X + 1$
- $X^5 + 1$
- $X^4 + 1$
- $X^4 + 4$

b) Show that the following polynomials are irreducible over $\mathbb{Q}$:

- $3X^4 - 8X^3 + 6X^2 - 4X + 6$
- $\frac{2}{3}X^5 + \frac{1}{2}X^4 - 2X^2 + \frac{1}{2}$
- $\frac{1}{5}X^4 - \frac{1}{3}X^3 - \frac{2}{3}X + 1$