Math 31: Topics in Algebra
Summer 2019 - Problem Set 8

Due: Wednesday, August 21

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#1 Use Eisenstein's criterion with the appropriate change of variable to show that each of the following polynomials is irreducible over $\mathbb{Q}$:

a) $X^4 + 2X^2 - 1$

b) $X^3 - 3X + 1$

#2 Use the natural homomorphism $\overline{h}_n : \mathbb{Z}[X] \to \mathbb{Z}_n[X]$ to show that the polynomial $X^4+7X^3+14X^2+3$ has no roots in $\mathbb{Q}$ (find the correct n!). How else might you show there are no rational roots of this polynomial?

#3

a) Show that $2+3i$ is algebraic over $\mathbb{Q}$ by finding its minimal polynomial $p(X)$. How do you know $p(X)$ is irreducible?

b) Find a monic irreducible polynomial $p(X)$ over $\mathbb{Q}$ such that $\mathbb{Q}[X]/\langle p(X)\rangle \cong \mathbb{Q}(1+\sqrt{2})$.

c) Find a complex number a such that $p(X) = X^4 + 2X^2 - 1$ is the minimal polynomial of a over $\mathbb{Q}$.

#4 Write down the 8 elements of $\mathbb{Z}_2[X]/\langle X^3 + X + 1\rangle$. Write down its multiplication table and confirm every element (except 0) has an inverse.

#5

a) Prove that $\{[a, b, c]^t : 2a - 3b + c = 0\}$ is a subspace of $\mathbb{R}^3$. Find its dimension and a basis for it as a subspace.

b) Prove that if V is a vector space over F with basis $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}\}$, then for any nonzero scalars $k_1, k_2, \ldots, k_n \in F$, $\{k_1\overrightarrow{a_1}, k_2\overrightarrow{a_2}, \ldots, k_n\overrightarrow{a_n}\}$ is also a basis for V.

c) Let U and V be vector spaces over F and $h : U \to V$ a linear transformation. Prove that $\mathrm{Ker}(h)$ is a subspace of U (the *null space*) and $h(U)$ is a subspace of V (the *range space*).
