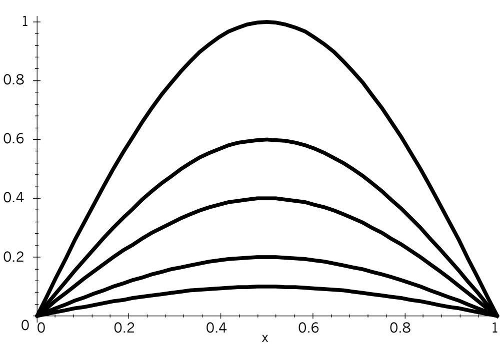
Mathematics 33

Homework Assignment #4

Due May 3

1. (p. 17, 1, 5 points). If the initial temperature of the rod were $u(x, 0) = \sin \pi x$, $0 \le x \le 1$ and if the BCs were $u(0, t) = 0, u(1, t) = 0$ what would be the behavior of the rod temperature $u(x, t)$ for later values of time.

Solution. The steady state solution of $u_t = \alpha^2 u_{xx}$ is when u does not depend on t that implies $u_t = 0$. Thus, the steady state solution satisfies the ODE $\alpha^2 u_{xx} = 0$ and therefore the steady state solution $\overline{u} = \overline{u}(x)$ must be a linear function of x with zero end-points. This implies that the steady state solution is zero, $\overline{u}(x) = 0, 0 \le x \le 1$ because $\overline{u}(0) = \overline{u}(1) = 0$. Since $u_{xx} = (\sin \pi x)'' = -\pi^2 \sin \pi x < 0$ the temperature within the rod will drop gradually until the steady state, see Figure below.

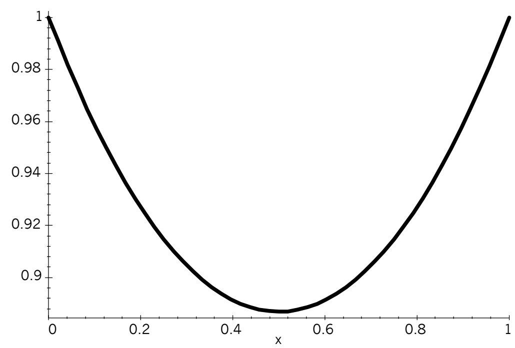


Temperature behavior in the rod starting from $\sin(\pi x)$. The temperature drops gradually until 0 because the second derivative of u is negative (u is concave).

2. (p. 17, 3, 5 points). Suppose a metal rod loses heat across the lateral boundary according to the equation $u_t = \alpha^2 u_{xx} - \beta u, 0 < x < 1$, and suppose we keep the ends of the rod at $u(0, t) = u(1, t) = 1$. Find the steady state temperature of the rod (graph it). Where is heat flowing in this problem?

Solution. As in the previous problem we find the steady state solution as the solution to the ODE $0 = \alpha^2 u_{xx} - \beta u$; we notice that this is ODE. To solve this ODE we introduce the characteristic (algebraic) equation $\alpha^2 z^2 - \beta = 0$ with the roots $z_{1,2} = \pm\sqrt{\beta/\alpha^2}$. Denoting $\lambda = \sqrt{\beta}/\alpha$ the solution to $0 = \alpha^2 u_{xx} - \beta u$ is $\overline{u} = \overline{u}(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$, the steady state temperature in the rod, $0 < x < 1$. Constants C_1, C_2 are found from BCs: $C_1 + C_2 = 1, C_1 e^{\lambda} + C_2 e^{-\lambda} = 1$. Solving this system for C_1, C_2 we obtain $C_1 = 1/(1 + e^{\lambda}), C_2 = e^{\lambda}/(1 + e^{\lambda})$. Finally, the steady state solution is

$$\overline{u}(x) = \frac{1}{1 + e^{\lambda}} \left(e^{\lambda x} + e^{\lambda(1-x)} \right)$$



Steady state solution when the rod loses heat across the lateral boundary.

The heat flows in at the ends of the rod as follows from BCs $u(0, t) = u(1, t) = 1$. From graph we see that at the ends temperature is 1^o but otherwise it is less due to the heat loss at the ends. Minimum temperature occurs in the middle of the rod.