Mathematics 33

Homework #6

Due May 17

1. (p. 41, 4, 5 points). Find the solution to the following problem: PDE: $u_t = u_{xx}$ for $0 < x < 1$. BCs: $u(0,t) = u(1,t) = 0$, $t > 0$. IC: $u(x,0) = 1$, $0 \le x \le 1$.

Solution. The solution is

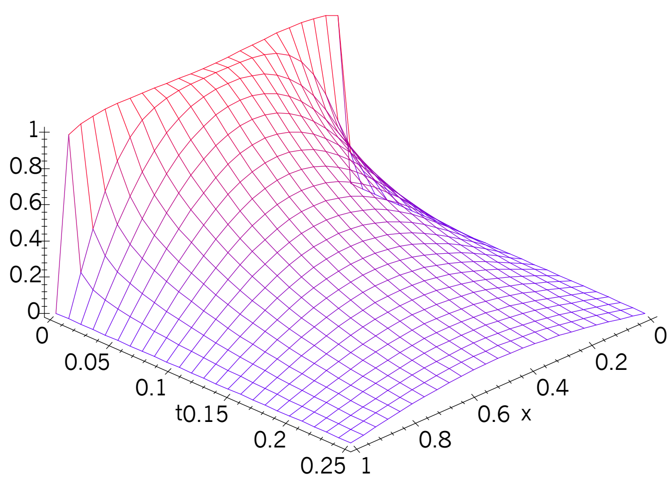
$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \exp\left(-(n\pi)^2 t\right)$$

where

$$A_n = 2\int_0^1 u(x,0)\sin(\pi n x)dx = 2\int_0^1 \sin(\pi n x)dx = \begin{cases} 0 \text{ if } n\text{=even} \\ \frac{4}{\pi n} \text{ if } n\text{=odd} \end{cases}.$$

Thus,

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)\pi x) \exp\left(-((2n-1)\pi)^2 t\right).$$



3D plot of function $u(x,t)$ where $x \in (0,1)$ and $t \in (0,.25)$

2. (p. 48, 2, 5 points). Transform PDE: $u_t = u_{xx}, 0 < x < 1; u(0,t) = 0, u(1,t) = 1, t > 0; u(x,0) = x^2, 0 \le x \le 1$ to zero BCs and solve the new problem. What will the solution to this problem look like for different values of time? Does the solution agree with your intuition? What is the steady state solution? What does the transient solution look like?

Solution. The steady-state solution has the form $\overline{u}(x) = x$ so that we can represent $u(x,t) = \overline{u}(x) + U(x,t)$ where $U(x,t)$ is the transient solution. In terms of $U(x,t) = u(x,t) - \overline{u}(x) = u(x,t) - x$

we have $U_t = U_{xx}, 0 < x < 1; U(0,t) = U(1,t) = 0; U(x,0) = x^2 - x, 0 \le x \le 1$. The general solution for U is

$$U(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \exp\left(-(n\pi)^2 t\right)$$

where

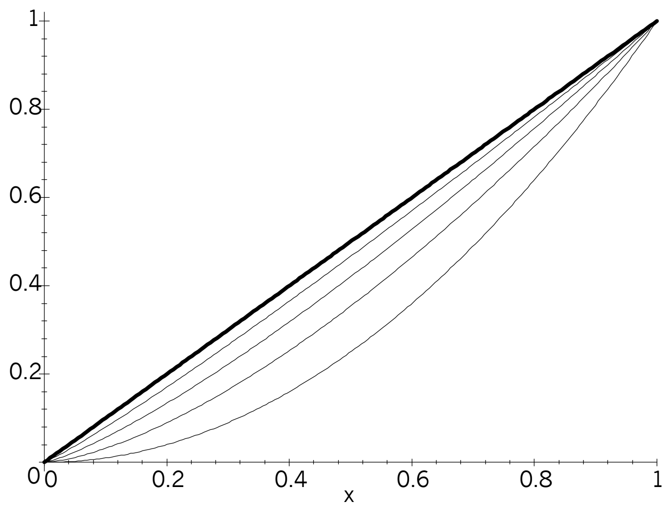
$$A_n = 2\int_0^1 (x^2 - x)\sin(\pi n x)dx = 4\frac{\cos n\pi - 1}{n^3\pi^3} = -\frac{8}{n^3\pi^3}\begin{cases} 0 \text{ if } n\text{=even} \\ 1 \text{ if } n\text{=odd} \end{cases} .$$

Therefore,

$$U(x,t) = -\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x)\exp\left(-((2n-1)\pi)^2 t\right),$$

and doing back transformation we finally obtain

$$u(x,t) = x - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin((2n-1)\pi x)\exp\left(-((2n-1)\pi)^2 t\right).$$



Solutions for different time t. The temperature at every point x increases with time because the second derivative of $\phi(x) = x^2$ is positive. The steady-state temperature (bold) is x.

3. (p. 61, 1, 5 points). Solve the diffusion problem PDE: $u_t = u_{xx} - u_x, 0 < x < 1, 0 < t < \infty$. BCs: $u(0,t) = u(1,t) = 0$. IC: $u(x,0) = e^{x/2}, 0 \le x \le 1$ by transforming it into an easier problem. What does the solution look like? We could interpret this problem as describing the concentration $u(x,t)$ in a moving medium (moving from left to right with velocity $n = 1$) where the concentration at the *ends* of the medium are kept zero (by some filtering device and initial *concentration* is $e^{x/2}$. Does your solution agree with this interpretation?

Solution. We introduce a new function $w(x,t)$ such that $u(x,t) = e^{(x-t/2)/2} w(x,t)$. Then,

$$\begin{aligned}
u_t &= -\frac{1}{4} e^{(x-t/2)/2} w + e^{(x-t/2)/2} w_t, \\
u_x &= \frac{1}{2} e^{(x-t/2)/2} w + e^{(x-t/2)/2} w_x, \\
u_{xx} &= \frac{1}{4} e^{(x-t/2)/2} w + e^{(x-t/2)/2} w_x + e^{(x-t/2)/2} w_{xx},
\end{aligned}$$

and the original PDE $u_t = u_{xx} - u_x$ transforms into

$$\begin{aligned}
& -\frac{1}{4} e^{(x-t/2)/2} w + e^{(x-t/2)/2} w_t \\
= \;& \frac{1}{4} e^{(x-t/2)/2} w + e^{(x-t/2)/2} w_x + e^{(x-t/2)/2} w_{xx} - \frac{1}{2} e^{(x-t/2)/2} w - e^{(x-t/2)/2} w_x.
\end{aligned}$$

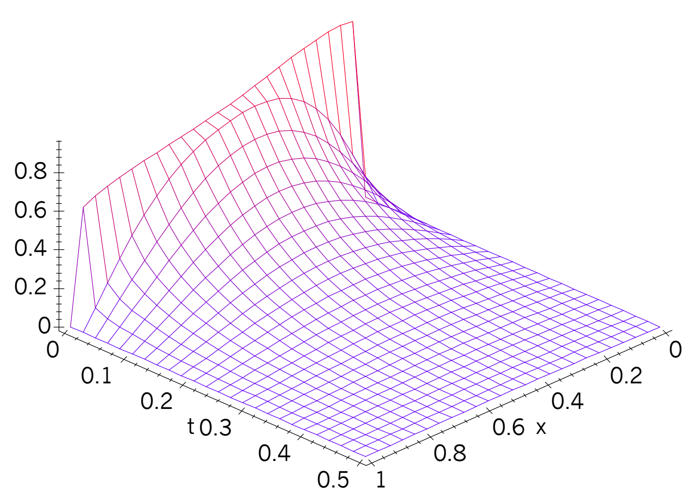
After a series of cancellations we come to the PDE $w_t = w_{xx}$ with BCs $w(0,t) = w(1,t) = 0$ and IC $w(x,0) = 1$. But the solution to this problem was derived in the previous homework (p.41, 4),

$$w(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)\pi x) \exp\left(-((2n-1)\pi)^2 t\right).$$

Thus the solution to the original PDE is

$$\begin{aligned}
u(x,t) &= e^{-(x-t/2)/2} w(x,y) \\
&= \frac{4}{\pi} e^{-(x-t/2)/2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)\pi x) \exp\left(-((2n-1)\pi)^2 t\right).
\end{aligned}$$

The 3D graph of this function is shown below (note: the text has a wrong answer to this problem, exponential part in the sum is omitted).



4. (p. 80, 3, 5 points). Solve by means of the sine or cosine transform PDE $u_t = \alpha^2 u_{xx}, x > 0$ with BC $u_x(0,t) = 0$ and IC

$$u(x,0) = H(1-x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases} .$$

What does the graph of the solution look like for various values of time?

Solution. We apply cosine transform, $U = \mathcal{F}_c(u)$ which gives $\mathcal{F}_c(u_t) = U'(t)$. For the second order derivative we obtain (see p. 76, formula 4 of our text)

$$\mathcal{F}_c(u_{xx}) = -\frac{2}{\pi}U'(0) - \omega^2 U.$$

For this problem $U'(0) = \mathcal{F}_c(u_x(0,t)) = \mathcal{F}_c(0) = 0$, in fact this justifies why cosine transform is used. Thus, after cosine transform we come to ODE

$$\frac{dU}{dt} = -\omega^2\alpha^2 U.$$

Now we find $U(0)$ using IC $u(x,0) = H(1-x)$. By definition, $U(0) = \mathcal{F}_c(u(x,0)) = \mathcal{F}_c(H(1-x))$. The latter formula is provided in our text (formula 6, p. 402): $\mathcal{F}_c(H(1-x)) = \frac{2}{\pi\omega}\sin(\omega)$. Thus, we have

$$U(\omega,t) = U(0)e^{-\omega^2\alpha^2 t} = \frac{2}{\pi\omega}\sin(\omega)e^{-\omega^2\alpha^2 t}.$$

We use inverse cosine transform to find $u(x,t)$ from $U(\omega,t)$ based on the general formula

$$u(x,t) = \int_0^\infty U(\omega,t)\cos(\omega x)d\omega$$

which leads to the answer

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \omega^{-1}\sin(\omega)e^{-\omega^2\alpha^2 t}\cos(\omega x)d\omega.$$

Note: The text gives a wrong answer: α^2 is omitted. It is possible to take sine transform then the answer is as in the text.