Math 35: Real analysis
Winter 2018 - Homework 1

Total: 20 points — Return date: Wednesday 01/10/18

Chapter 1.1

keywords: *rational and irrational numbers.*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 2. *(2 points)* Prove that there is a rational number between any two distinct rational numbers.

Solution: For $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, where $\frac{a}{b} < \frac{c}{d}$ we take the mean value $\frac{1}{2} \cdot \left(\frac{a}{b} + \frac{c}{d}\right) \in \mathbb{Q}$. Then

$$\frac{a}{b} = \frac{1}{2}\left(\frac{a}{b} + \frac{a}{b}\right) < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{1}{2}\left(\frac{c}{d} + \frac{c}{d}\right) = \frac{c}{d}.$$

exercise 3. *(2 points)* Convert each rational number into decimal numbers.
Solution: a) $\frac{8}{27} = 0.\overline{296}$ b) $\frac{4}{21} = 0.1\overline{90476}$ c) $\frac{5}{19} = 0.\overline{263157894736842105}$.

exercise 5. *(2 points)* Find the millionth digit in the decimal expansion of $\frac{2}{7}$.
Solution: We have that $\frac{2}{7} = 0.\overline{285714}$.
We have $1000000 = 166666 \cdot 6 + 4$. So the millionth digit is 7.

exercise 7. *(3 points)* Prove that the sum of a rational and an irrational number is irrational.
Solution: We have to show that if $p \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$ then $x + q = r \in \mathbb{R} \setminus \mathbb{Q}$.
We prove this fact by contradiction: Suppose that $r \in \mathbb{Q}$, then $r - q \in \mathbb{Q}$, hence

$$x = r - q \in \mathbb{Q},$$

which is a contradiction.

exercise 8. *(3 points)* Prove that the product of a non-zero rational number and an irrational number is irrational.
Solution: We have to show that if $p \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$ then $x \cdot q = r \in \mathbb{R} \setminus \mathbb{Q}$.
We prove this fact by contradiction: Suppose that $r \in \mathbb{Q}$, then $r \cdot \frac{1}{q} \in \mathbb{Q}$, hence

$$x = r \cdot \frac{1}{q} \in \mathbb{Q},$$

which is a contradiction.

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exercise 11. *(4 points)* Let n be a positive integer that is not a perfect square. Prove that $\sqrt{n}$ is irrational.
Solution: The proof uses similar ideas as the proof for the case $n = 2$: We know that n is a positive integer that is not a perfect square. Suppose that

$$\sqrt{n} = \frac{a}{b} \in \mathbb{Q}, \quad \text{where} \quad a, b \in \mathbb{Z}$$

and $\frac{a}{b}$ is a reduced fraction. We know that n can be factored into primes and at least one of those primes, say $p|n$ occurs oddly many times, as otherwise n would be a perfect square. We also know that

$$n = \frac{a^2}{b^2} \quad \text{or} \quad n \cdot b^2 = a^2.$$

But then p shows up oddly many times as a factor on the left hand side, but evenly many times on the right hand side. A contradiction.

exercise 13. *(4 points)* Prove that $r = \sqrt{n-1} + \sqrt{n+1}$ is irrational for every positive integer n.
Solution: For $n = 1$ we know from the lecture that $r = \sqrt{1-1} + \sqrt{1+1} = \sqrt{2}$ is irrational.
For $n \geq 2$ we argue by contradiction: Suppose that $\sqrt{n-1} + \sqrt{n+1} \in \mathbb{Q}$. Then

$$r^2 = (\sqrt{n-1} + \sqrt{n+1}) \cdot (\sqrt{n-1} + \sqrt{n+1}) \in \mathbb{Q}\,.$$

But then

$$r^2 = (n-1) + (n+1) + 2\sqrt{(n+1)(n-1)} = 2n + \sqrt{n^2-1} \in \mathbb{Q}\,.$$

By **exercise 7.** we know that

$$2n + \sqrt{n^2-1} \in \mathbb{Q} \Leftrightarrow \sqrt{n^2-1} \in \mathbb{Q}\,.$$

By **exercise 11.** $\sqrt{n^2-1} \in \mathbb{Q}$ if and only if $n^2 - 1$ is a perfect square.
However, we know that for all $n \geq 2$: $n^2 - 1$ is not a perfect square, as the difference between two perfect squares is at least $3 = 2^2 - 1^2$.
Hence $r^2 = 2n + \sqrt{n^2-1} \notin \mathbb{Q}$ and our assumption that $r = \sqrt{n-1} + \sqrt{n+1} \in \mathbb{Q}$ must be wrong. Hence $r \in \mathbb{R} \setminus \mathbb{Q}$ for every positive integer n.