Math 35: Real analysis
Winter 2018 - Homework 2

Total: 20 points — Return date: Wednesday 01/17/18

Chapter 1.2

keywords: *absolute value, inequalities, geometric sum.*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 4. *(5 points)* Prove **Theorem 1.4**
Solution: We recall:
Theorem 1.4 The absolute value of a function has the following properties: for all $a, b, c \in \mathbb{R}$

a) $-|a| \leq 0 \leq |a|$.

b) $|a| \cdot |b| = |a \cdot b|$.

c) $|-a| = |a|$.

d) Let $c > 0$. Then $|a| < c \Leftrightarrow -c < a < c$ and $|a| \leq c \Leftrightarrow -c \leq a \leq c$.

proof of a) Recall that

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \phantom{\text{if}} \quad a < 0. \end{cases}$$

By **Def. 1.3.3.** we have:

$$a < 0 \Leftrightarrow 0 - a > 0 \Leftrightarrow -a > 0 \;\; \text{and} \;\; -a < 0 \Leftrightarrow 0 - (-a) > 0 \Leftrightarrow a > 0.(**)$$

For our proof we distinguish three cases:

i.) $a = 0:$ We recall that $a = 0 \Leftrightarrow |a| = 0$. By the definition of $|a|$ we have $-|0| = 0 = |0|$ and our statement is true.

ii.) $a < 0:$ By the definition of $|a|$ we have $|a| = -a$, hence $-|a| = -(-a) = a$ and we have $-|a| = a \Rightarrow -|a| \leq a$.
Furthermore $a < 0$ and $0 < |a|$. Hence by **Def. 1.2.2** $a < |a|$, hence $a \leq |a|$.

iii.) $a > 0:$ By the definition of $|a|$ we have $|a| = a$, hence we also have $|a| \leq a$. Furthermore we know that $a = |a| > 0 \Leftrightarrow 0 < -|a|$. Hence by **Def. 1.2.2** $-|a| < a$, hence $-|a| \leq a$.

As by **Def. 1.2.1** these are all possible cases, the inequality $-|a| \leq a \leq |a|$ is always true.

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proof of b) Show that $|a \cdot b| = |a| \cdot |b|$. The case where $a = 0$ or $b = 0$ is trivial. We look at all remaining possible cases for the sign of a and b. We summarize the results in two tables:

$\|a\| \cdot \|b\|$	$a > 0$	$a < 0$
$b > 0$	$a \cdot b$	$(-a) \cdot b = -(a \cdot b)$
$b < 0$	$a \cdot (-b) = -(a \cdot b)$	$(-a) \cdot (-b) = a \cdot b$

$\|a \cdot b\|$	$a > 0$	$a < 0$
$b > 0$	$a \cdot b$	$-(a \cdot b)$
$b < 0$	$-(a \cdot b)$	$a \cdot b$

For the second table we have to gather some information about the sign of $a \cdot b$:

i.) $b > 0$ **and** $a > 0$: by **Def. 1.3.2** we have that $a \cdot b > 0$.

ii.) $b > 0$ **and** $a < 0$: by (**) we have that $-a > 0$ and hence $(-a) \cdot b = -(a \cdot b) > 0$ by **Def. 1.3.2**. But again by (**) this implies that $-(-(a \cdot b)) = a \cdot b < 0$.

iii.) $b < 0$ **and** $a > 0$: the same arguments imply that $a \cdot b < 0$.

iv.) $b < 0$ **and** $a < 0$: again by (**) $-a > 0$ and $-b > 0$. Hence by **Def. 1.3.2** $(-a) \cdot (-b) = a \cdot b > 0$.

In total we get from the defintion of $|x|$ that $|a| \cdot |b| = |a \cdot b|$.

proof of c) By b) and the defintion of a field we have that

$$|a| = |-1| \cdot |a| = |(-1) \cdot a| = |-a|.$$

proof of d) We only prove the first inequality. The second follows in a similar fashion.
"$\Rightarrow$" We know that $c > 0$ and $|a| < c$.

i.) $a > 0$: then $|a| = a$ and $0 < a < c$. By (**) we have that $-c < 0$, hence by **Def. 1.2.2** $-c < a$. In total we have that $-c < a < c$.

ii.) $a < 0$: As $c > 0$, then by **Def. 1.2.2** we have that $a < c$. We also know that $|a| = -a < c$ or $c - a > 0$ By (**) this implies that $a - c < 0$ or $-c < a$. In total $-c < a < c$.

"$\Leftarrow$" If $a > 0$ then $|a| = a$ and as $a < c$ there is nothing to prove.
If $a < 0$ then $|a| = -a$. As $-c < a \Leftrightarrow 0 < a - c$ by **Def. 1.3.3**. Then by (**) we have that $c - a < 0$ or $-a = |a| < c$.

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exercise 5. *(3 points)* Under which conditions does equality hold for the triangle and reverse triangle inequality:
Solution:

i.) Triangle inequality: We have $|a+b| = |a| + |b|$. To answer this question we can use that $|x| = \sqrt{x^2}$:

$$|a+b| = \sqrt{(a+b)^2} = \sqrt{a^2} + \sqrt{b^2} = |a| + |b|$$

As both sides are positive numbers, we can take the square on both sides and get

$$\sqrt{(a+b)^2} = \sqrt{a^2} + \sqrt{b^2} \Leftrightarrow (a+b)^2 = (|a|+|b|)^2 \Leftrightarrow a^2+b^2+2ab = a^2+b^2+2|a|\cdot|b| \Leftrightarrow a\cdot b = |a|\cdot|b|.$$

Hence we have equality if $a \cdot b = |a| \cdot |b|$ or a and b have the same sign. It is easy to check that this condition implies that the triangle inequality is true.

ii.) Reverse triangle inequality: We have $|a-b| = ||a| - |b||$. We use again that $|x| = \sqrt{x^2}$:

$$|a-b| = \sqrt{(a-b)^2} = \sqrt{(|a|-|b|)^2} = ||a| - |b|| \Leftrightarrow a^2 + b^2 - 2ab = a^2 + b^2 - 2|a| \cdot |b|.$$

Hence again we have equality if $a \cdot b = |a| \cdot |b|$ or a and b have the same sign.

exercise 8.a) *(2 points)* Let $n > 1$ be a positive integer and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be real numbers. Show that

$$|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|.$$

Solution: We use the triangle inequality several times:

$$|\sum_{k=1}^{n} a_k| = |a_1 + (\sum_{k=2}^{n} a_k)| \overset{\triangle\neq}{\leq} |a_1| + |\sum_{k=2}^{n} a_k| = |a_1| + |a_2 + (\sum_{k=3}^{n} a_k)| \overset{\triangle\neq}{\leq}$$
$$|a_1| + |a_2| + |\sum_{k=3}^{n} a_k| \overset{\triangle\neq}{\leq} \ldots \overset{\triangle\neq}{\leq} |a_1| + |a_2| \ldots |a_n| = \sum_{k=1}^{n} |a_k|.$$

exercise 20. *(3 points)* A ball has a bounce coefficient of 0.85. Suppose that the ball is dropped from a height of 10 feet. How far has it traveled when it hits the floor for the twelfth time?
Solution: We can make a table to see how far has traveled between the n-th and $n+1$-th touch.

drop	0-1	1-2	2-3	3-4	...	11-12
dist. traveled	10	$2 \cdot 0.85 \cdot 10$	$2 \cdot 0.85^2 \cdot 10$	$2 \cdot 0.85^3 \cdot 10$	...	$2 \cdot 0.85^{11} \cdot 10$

In total the ball has traveled

$$10 + 2 \cdot 10 \cdot \sum_{k=1}^{11} (0.85)^k \overset{geom.sum}{=} 10 + 20 \cdot \frac{1-(0.85)^{12}}{1-0.85} - 20 \simeq 104.37 \text{ ft.}$$

exercise 26. *(4 points)* Let $x, y > 0$ be two positive real numbers. For each of the following conditions on x and y find the maximum value for the product $x \cdot y$.

a) $4x + 9y = 36$.
Solution: With **Theorem 1.12** we have that

$$\frac{1}{2} \cdot (a_1 + a_2) \geq (a_1 \cdot a_2)^{\frac{1}{2}}.$$

and equality occurs if and only if $a_1 = a_2$. Setting $a_1 = 4x$ and $a_2 = 9y$ we obtain

$$18 = \frac{36}{2} \overset{4x+9y=36}{=} \frac{1}{2} \cdot (4x + 9y) \geq (4x \cdot 9y)^{\frac{1}{2}} = 6 \cdot (x \cdot y)^{\frac{1}{2}} \quad \text{hence} \quad \boxed{9 \geq xy.}$$

Furthermore equality is obtained if and only if $4x = a_1 = a_2 = 9y$. As $4x + 9y = 36$ equality occurs for $\boxed{x = 4.5}$ and $\boxed{y = 2.}$ For these values the maximum $\boxed{xy = 9}$ is attained.

b) $4x^2 + 9y^2 = 36$.
Solution: We use again that

$$\frac{1}{2} \cdot (a_1 + a_2) \geq (a_1 \cdot a_2)^{\frac{1}{2}}.$$

Setting $a_1 = 4x^2$ and $a_2 = 9y^2$ we obtain

$$18 = \frac{36}{2} \overset{4x^2+9y^2=36}{=} \frac{1}{2} \cdot \left(4x^2 + 9y^2\right) \geq \left(4x^2 \cdot 9y^2\right)^{\frac{1}{2}} = 6xy \quad \text{hence} \quad \boxed{3 \geq xy.}$$

Furthermore equality is obtained if and only if $4x^2 = a_1 = a_2 = 9y^2$. As $4x^2 + 9y^2 = 36$ equality occurs for $\boxed{x = \sqrt{4.5}}$ and $\boxed{y = \sqrt{2}.}$ For these values the maximum $\boxed{xy = 3}$ is attained.

c) $4x^2 + 9y = 36$.
Solution: This is a bit trickier. This time we can not use the AM-GM inequality with two values. The trick is to use the inequality with three values instead:

$$\frac{1}{3} \cdot (a_1 + a_2 + a_3) \geq (a_1 \cdot a_2 \cdot a_3)^{\frac{1}{3}}.$$

Setting $a_1 = 4x^2$ and $a_2 = a_3 = 4.5 \cdot y$ we obtain

$$12 = \frac{1}{3} \cdot \left(4x^2 + 4.5 \cdot y + 4.5 \cdot y\right) \geq \left(4x^2 \cdot 4.5 \cdot y \cdot 4.5 \cdot y\right)^{\frac{1}{3}} = (9xy)^{\frac{2}{3}} \quad \text{hence} \quad \boxed{\frac{8\sqrt{3}}{3} \geq xy.}$$

Furthermore equality is obtained if and only if $4x^2 = a_1 = a_2 = a_3 = 4.5y$. This means that $\boxed{x = \sqrt{3}}$ and $\boxed{y = \frac{8}{3}.}$ For these values the maximum $\boxed{xy = \frac{8\sqrt{3}}{3}}$ is attained.

exercise 34. *(3 points)* Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\sum_{k=1}^{n} |a_k| \leq \left(n \cdot \sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}.$$

Solution: We look at $\tilde{\mathbf{a}} = (|a_1|, |a_2|, \ldots, |a_n|)$ and $\mathbf{b} = (1, 1, \ldots, 1)$. Then the Cauchy-Schwarz inequality for $\tilde{\mathbf{a}}$ and $\mathbf{b}$ gives us

$$\left(\sum_{k=1}^{n} |a_k|\right)^2 = |\tilde{\mathbf{a}} \bullet \mathbf{b}|^2 \leq |\tilde{\mathbf{a}}|^2 \cdot |\mathbf{b}|^2 = \left(\sum_{k=1}^{n} a_k^2\right) \cdot n.$$

We obtain the above inequality by taking the root on both sides.