Math 35: Real analysis
Winter 2018 - Homework 3

Total: 20 points | Return date: Wednesday 01/24/18

Chapter 1.3 and 1.4

keywords: *infima and suprema, countable and uncountable sets*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 10 *(3 points)* Prove that for the open interval (a, b) the infimum is a and the supremum is b.

Solution: We start with the supremum: We have to show that

1.) b is an upper bound of (a, b), i.e. $b \geq x$ for all $x \in (a, b)$.

2.) For every $\beta < b$ there is an $x \in (a, b)$, such that $\beta < x$.

1.) As $(a, b) = \{x \in \mathbb{R}, a < x < b\}$ we know that b is an upper bound for (a, b).
2.) It remains to show that for all $\beta < b$ we have that β is not an upper bound:
As $\beta < b$ we have that $b - \beta = \epsilon > 0$. By **Theorem 1.17.4** we know that there is a positive integer $n \in \mathbb{N}$, such that

$$\epsilon < \frac{1}{n} \Leftrightarrow -\epsilon < -\frac{1}{n}.$$

Hence

$$\beta = b - (b - \beta) = b - \epsilon < b - \frac{1}{n} \in (a, b).$$

Hence there is an element $b - \frac{1}{n} \in (a, b)$, which is strictly greater than β. Therefore β can not be an upper bound. In total we conclude that b is the supremum of (a, b).
In a similar fashion it is clear that a is a lower bound. Furthermore by similar arguments as for the supremum any $\alpha > a$ can not be a lower bound for (a, b). Hence a is the infimum of (a, b).

exercise 16 *(2 points)* For **Theorem 1.17.** show that 1. $\Rightarrow$ 2. Recall:

Theorem 1.17. The following statements are equivalent:

1. If $a, b > 0$ then there is a positive integer $n \in \mathbb{N}$, such that $n \cdot a > b$.
2. The set $\mathbb{N}$ of positive integers is not bounded above.
3. For each $x \in \mathbb{R}$ there is an integer $n \in \mathbb{Z}$, such that $n \leq x < n + 1$.
4. For each $x \in \mathbb{R}, x > 0$ there is a positive integer $n \in \mathbb{N}$, such that $\frac{1}{n} < x$.

Solution: 1. $\Rightarrow$ 2.: By contradiction: Suppose $\mathbb{N}$ is bounded from above. Then there is $b \in \mathbb{R}$, such that $m \geq b$ for all $m \in \mathbb{N}$. But by 1. with $a = 1$ there is $n \in \mathbb{N}$, such that $n \cdot 1 = n > b$, a contradiction.

exercise 25. *(5 points)* Show that each real number $x \in [0, 1]$ has a **binary expansion**.

Solution: Let $x \in [0, 1]$. Let b_0 be the largest integer, such that $b_0 \leq x$. Clearly $b_0 \in \{0, 1\}$. Let b_1 be the largest integer, such that

$$b_0 + \frac{b_1}{2} \leq x \leq 1.$$

Note that $b_1 \in \{0, 1\}$ as for $x \neq 1$ we have that $b_0 = 0$ and

$$b_0 + \frac{b_1}{2} \leq x < 1 \quad \text{hence} \quad b_1 < 2. \quad (*)$$

Suppose that for some integer $n \geq 2$ integers $b_0, b_1, \ldots, b_{n-1}$ have been chosen this way. Let b_n be the largest integer, such that

$$b_0 + \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} \leq x \leq 1.$$

We prove that $b_n \in \{0, 1\}$ by induction for $x \neq 1$ i.e. $b_0 = 0$.

1.) Induction start ($n = 1$): This was shown in (*).
2.) Induction step ($n \to n + 1$): We know that

$$b_k \in \{0, 1\} \quad \text{for all} \quad k \in \{1, 2, \ldots, n\} \quad \textbf{(Induction hypothesis)} \quad (**)$$

We have to show that this is true for $k \in \{1, 2, \ldots, n + 1\}$. By definition we have

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} + \frac{b_{n+1}}{2^{n+1}} \leq x.$$

By definition b_n was the largest integer, such that $\sum_{k=1}^{n} \frac{b_k}{2^k} \leq x$. If $b_{n+1} \geq 2$ then this contradicts the fact that b_n is maximal. Hence $b_{n+1} < 2$. This proves our statement. $\square$
By induction we can define b_n for all integers $n \in \mathbb{N}$. We define the set of numbers

$$B := \{b_0 + \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n}, n \in \mathbb{N} \cup \{0\}\} = \{b_0, b_0 + \frac{b_1}{2}, b_0 + \frac{b_1}{2^1} + \frac{b_2}{2^2}, \ldots, \sum_{k=0}^{n} \frac{b_k}{2^k}, \ldots\}$$

The set B has the upper bound x and by the Completeness Axiom this implies that a supremum $\sup(B) = \beta$ exists. Suppose $\beta < x$. Then by **Theorem 1.17.4** there is an $m \in \mathbb{N}$, such that

$$\frac{1}{2^m} \leq \frac{1}{m} < x - \beta \quad \text{or} \quad \beta + \frac{1}{2^m} < x. \quad (***)$$

By the maximality of b_m and as β is the supremum of B we have that:

$$x < \underbrace{b_0 + \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots + \frac{b_m}{2^m}}_{\in B} + \frac{1}{2^m} \leq \beta + \frac{1}{2^m} \overset{(***)}{<} x,$$

a contradiction. In other words

$$x = b_0, b_1\ b_2\ b_3 \ldots b_n \ldots = b_0 + \frac{b_1}{2^1} + \frac{b_2}{2^2} + \ldots + \frac{b_n}{2^n} \ldots.$$

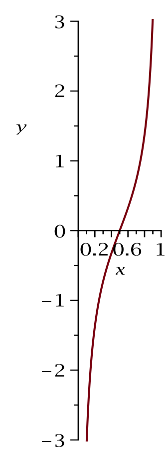
exercise 1. d), f) *(3 points)* For each pair of sets find a one-to-one correspondence between them.

d) $[1,2]$ and $[1,6]$.
Solution: $f : [1,2] \to [1,6], x \mapsto f(x) := 5(x-1)+1 = 5x-4$. The inverse map is $f^{-1}(x) = \frac{x+4}{5}$. We check that this is indeed the inverse map i.e.

$$f \circ f^{-1}(x) = x \quad \text{and} \quad f^{-1} \circ f(x) = x.$$

As $f(1) = 1$ and $f(2) = 6$ we have that $f([1,2]) = [1,6]$.

f) $\mathbb{R}$ and $(0,1)$.
Solution: $g : (0,1) \to \mathbb{R}, x \mapsto g(x) := \tan(\pi \cdot x - \frac{\pi}{2})$. We know that $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a strictly increasing function (derivative test). This implies that *tan* is injective. As $\tan(-\frac{\pi}{2}) = -\infty$ and $\tan(\frac{\pi}{2}) = \infty$ the function is also surjective, hence bijective. By scaling and shifting this function we obtain $g : (0,1) \to \mathbb{R}$ (see below).



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exercise 7. *(3 points)* Prove that union $A \uplus B$ of two disjoint countably infinite sets A and B is countably infinite by finding a one-to-one correspondence of the union with $\mathbb{N}$.

Solution: Let E^+ be the even positive integers and O^+ be the odd positive integers. Similar as in the examples from **Lecture 6** we can establish the one to one correspondences

$$E^+ \leftrightarrow \mathbb{N} \leftrightarrow A \text{ and } O^+ \leftrightarrow \mathbb{N} \leftrightarrow B.$$

Let $f : E^+ \to A$ and $g : O^+ \to B$ be the bijective maps between these sets. We set

$$h : \mathbb{N} = E^+ \uplus O^+ \to A \uplus B, n \to h(n) := \begin{cases} g(n) & \text{if} \quad n \text{ odd} \\ f(n) & \quad\;\; n \text{ even} \end{cases}.$$

Then h is a bijective map. This shows that $A \uplus B$ is countably infinite.

exercise 18. c) *(4 points)* Let A be the set of all countably infinite sequences of 0's and 1's.

Example: $1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, \ldots$

Prove that $\mathcal{P}(\mathbb{N})$, the collection of all subsets of positive integers is uncountable by establishing a one-to-one correspondence between $\mathcal{P}(\mathbb{N})$ and A.

Solution: For each element S in $\mathcal{P}(\mathbb{N})$ we construct a binary sequence $(b_k^S)_{k\geq 1}$ in the following way. We set

$$b_k^S = \begin{cases} 1 & \text{if} \quad k \in S \\ 0 & \quad\;\; k \notin S. \end{cases}$$

Then the map

$$f : \mathcal{P}(\mathbb{N}) \to A, S \mapsto f(S) := (b_k^S)_k$$

is a bijective map from $\mathcal{P}(\mathbb{N})$ to A.
In a same way as in the lecture for the numbers in $[0, 1]$, we can use the diagonal argument to show that A is uncountable.
We can also argue that there is a surjective map from A to $[0, 1]$ by writing each real number in $[0, 1]$ with its binary expansion. By removing multiple mappings to elements in $[0, 1]$ we can find a subset B of A that is in one-to-one correspondence with $[0, 1]$. This implies again that A is uncoutable as it contains the uncountable set B.
Therefore, as A is uncountable, so is $\mathcal{P}(\mathbb{N})$.