Math 35: Real analysis
Winter 2018 - Homework 5

Total: 20 points Return date: Wednesday 02/07/18

Chapter 6.1 - 6.3

keywords: *infinite series*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

Chapter 6.1

exercise 1. c) *(3 points)* Find a simple expression for the sequence of partial sums. Then find out whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right).$$

Solution: We first note that $\ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln(k)$. Hence

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} \ln(k+1) - \ln(k) = \sum_{k=1}^{n} \ln(k+1) - \sum_{k=1}^{n} \ln(k) \\ &= (\ln(2) + \ln(3) + \ldots + \ln(n) + \ln(n+1)) \\ &\quad -(\ln(1) + \ln(2) + \ln(3) + \ldots + \ln(n)) = \ln(n+1) - \ln(1) = \ln(n+1). \end{aligned}$$

The series converges if and only if the sequence $(S_n)_n$ converges. But

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty.$$

Hence the sequence and therefore the series diverge.

exercise 3. *(4 points)* Let $p \in \mathbb{N}$ be a fixed natural number. Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+p)}.$$

Solution: Idea: We would like to relate the product $\frac{1}{k(k+p)}$ to the sum $\frac{a}{k} + \frac{b}{(k+p)}$ to (hopefully) turn the series into a telescoping sum. So we try to find constants A and B, such that

$$\boxed{\frac{0 \cdot k + 1}{k(k+p)}} = \frac{1}{k(k+p)} = \frac{A}{k} + \frac{B}{k+p} = \frac{A \cdot (k+p) + B \cdot k}{k(k+p)} = \boxed{\frac{(A+B) \cdot k + A \cdot p}{k(k+p)}}.$$

As k varies this is only possible if $A = -B$ and $1 = A \cdot p$, hence $A = \frac{1}{p}$ and $B = -\frac{1}{p}$. We obtain

$$\frac{1}{k(k+p)} = \frac{1}{p} \cdot \left(\frac{1}{k} - \frac{1}{k+p} \right).$$

Hence

$$\begin{aligned}
\boxed{p \cdot S_n} &= p \cdot \sum_{k=1}^{n} \frac{1}{k(k+p)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+p} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+p} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{p} \right) + \left(\frac{1}{p+1} + \frac{1}{p+2} + \ldots + \frac{1}{n} \right) \\
&\qquad - \left(\frac{1}{p+1} + \frac{1}{p+2} + \ldots + \frac{1}{n} \right) - \left(\frac{1}{n+1} + \ldots + \frac{1}{n+p} \right) \\
&= \sum_{k=1}^{p} \frac{1}{k} - \sum_{k=1}^{p} \frac{1}{n+k}.
\end{aligned}$$

The series converges if and only if the sequence $(S_n)_n$ converges. We have

$$\lim_{n\to\infty} S_n = \frac{1}{p} \cdot \lim_{n\to\infty} \left(\sum_{k=1}^{p} \frac{1}{k} - \sum_{k=1}^{p} \frac{1}{n+k} \right) = \frac{1}{p} \cdot \left(\sum_{k=1}^{p} \frac{1}{k} - \sum_{k=1}^{p} \underbrace{\lim_{n\to\infty} \frac{1}{n+k}}_{=0} \right) = \boxed{\frac{1}{p} \cdot \sum_{k=1}^{p} \frac{1}{k}}.$$

Hence the sequence and therefore the series converge.

Chapter 6.2

exercise 1. *(4 points)* Prove the Comparison test. We recall

Theorem 6.6 (Comparison test) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series, such that

$$a_k, b_k \geq 0 \;\text{ for all }\; k \in \mathbb{N} \;\text{ and }\; a_k \leq b_k \;\text{ for all }\; k \geq K \in N.$$

Then

a) If the series $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges.

b) If the series $\sum_{k=1}^{\infty} a_k$ diverges, then the series $\sum_{k=1}^{\infty} b_k$ diverges.

Solution: We recall that a series converges, if and only if its sequence of partial sums converges. Let $(S_n)_n = \left(\sum_{k=1}^{n} a_k\right)_n$ and $(T_n)_n = \left(\sum_{k=1}^{n} a_k\right)_n$ be the two sequences of partial sums. We first note that, as all elements of the sequence $(a_k)_k$ are non-negative that

$$S_{n+1} - S_n = a_{n+1} \geq 0 \Rightarrow S_{n+1} \geq S_n \quad \text{for all } \; n \in \mathbb{N}\,.$$

Hence $(S_n)_n$ is an increasing sequence and by **Theorem 2.10** of the book it converges if and only if it is bounded. Furthermore the same is true for the sequence $(T_n)_n$.
a) As the series $\sum_{k=1}^{\infty} b_k$ converges, we know that $(T_n)_n$ is bounded. This implies that, as $a_k \leq b_k$ for all $k \geq K$ that

$$0 \leq \sum_{k=K}^{\infty} a_k \leq \sum_{k=K}^{\infty} b_k \leq \sum_{k=1}^{\infty} b_k \leq B.$$

As $\sum_{k=1}^{K-1} a_k$ is a finite sum and therefore bounded from above by some $A \in \mathbb{R}$, we have

$$0 \leq \sum_{k=1}^{\infty} a_k = \underbrace{\sum_{k=1}^{K-1} a_k}_{\leq A} + \sum_{k=K}^{\infty} a_k \leq A + B.$$

Hence $\sum_{k=1}^{\infty} a_k$ is bounded and therefore the series converges.
b) In a similar fashion, if $\sum_{k=1}^{\infty} a_k$ diverges then it must be unbounded, as if it was bounded, it would converge by **Theorem 2.10**. Hence

$$\lim_{n\to\infty} S_n = \infty \quad \text{and} \quad \lim_{n\to\infty} S_n - \sum_{k=1}^{K-1} a_k = \lim_{n\to\infty} \sum_{k=K}^{n} a_k = \infty.$$

This means that $\sum_{k=K}^{\infty} a_k$ is unbounded and therefore also have

$$\infty = \lim_{n\to\infty} \sum_{k=K}^{n} a_k = \lim_{n\to\infty} \sum_{k=K}^{n} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} b_k$$

Hence the series $\sum_{k=1}^{\infty} b_k$ diverges.

exercise 8. *(3 points)* Find the sum of the given series.
b) **Solution:** For $\sum_{k=4}^{\infty} \frac{10\cdot 2^{k+1}}{5^{k-3}}$ we get

$$\begin{aligned}\sum_{k=4}^{\infty} \frac{10 \cdot 2^{k+1}}{5^{k-3}} &= 10 \cdot \sum_{k=4}^{\infty} \frac{2 \cdot 2^k}{\frac{1}{5^3} \cdot 5^k} = 2500 \cdot \sum_{k=4}^{\infty} \left(\frac{2}{5}\right)^k = 2500 \cdot \left(\sum_{k=\mathbf{0}}^{\infty} \left(\frac{2}{5}\right)^k - \sum_{k=0}^{3} \left(\frac{2}{5}\right)^k\right) \\ &\overset{geom.\ series}{=} 2500 \cdot \left(\frac{1}{1-\frac{2}{5}} - \left(1 + \frac{2}{5} + \frac{2^2}{5^2} + \frac{2^3}{5^3}\right)\right) = \boxed{\frac{320}{3}}.\end{aligned}$$

c) **Solution:** For $\sum_{k=1}^{\infty} \frac{3}{2^k} - \frac{2}{3^{k+1}}$ we get

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{3}{2^k} - \frac{2}{3^{k+1}} &= \sum_{k=1}^{\infty} \frac{3}{2^k} - \sum_{k=1}^{\infty} \frac{2}{3^{k+1}} = \left(\sum_{k=0}^{\infty} \frac{3}{2^k} - 3\right) - \left(\sum_{k=0}^{\infty} \frac{2}{3 \cdot 3^k} - \frac{2}{3}\right) \\ &\overset{geom.\ series}{=} \left(\frac{3}{1-\frac{1}{2}} - 3\right) - \left(\frac{2}{3} \cdot \frac{1}{1-\frac{1}{3}} - \frac{2}{3}\right) = 3 - \frac{1}{3} = \boxed{\frac{8}{3}}.\end{aligned}$$

Chapter 6.3

exercise 4. *(3 points)* Suppose that the series $\sum_{k=1}^{\infty} a_k$ converges absolutely. Prove that the series $\sum_{k=1}^{\infty} a_k^2$ converges.

Solution: We know that $\sum_{k=1}^{\infty} |a_k|$ converges and that $\boxed{\lim_{k\to\infty} a_k = 0}$ by **Theorem 6.2** of the book. This means that for $\epsilon = 1$ there is $N = N(\epsilon) \in \mathbb{N}$, such that

$$|a_k - 0| = |a_k| \leq 1 \quad \text{for all} \quad k \geq N$$

As $x^2 \leq |x|$ for all $x \in [0, 1]$ we have that

$$0 \leq a_k^2 \leq |a_k| \quad \text{for all} \quad k \geq N.$$

As $\sum_{k=1}^{\infty} |a_k|$ converges the **Comparison test** implies that the series $\sum_{k=1}^{\infty} a_k^2$ converges.

exercise 12. c) *(3 points)* Determine whether the series

$$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} \quad \text{converges.}$$

Solution: We try the **Ratio test**. As all a_k are positive, we have

$$\begin{aligned}\frac{|a_{k+1}|}{|a_k|} &= \frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{(k!)^2} = \frac{(2k)!}{(2k+2)!} \cdot \frac{((k+1)!)^2}{(k!)^2} = \frac{(2k)!}{(2k+2) \cdot (2k+1) \cdot (2k)!} \cdot \left(\frac{(k+1)!}{k!}\right)^2 \\ &= \frac{1}{(2k+2) \cdot (2k+1)} \cdot \left(\frac{(k+1) \cdot k!)}{k!}\right)^2 = \frac{(k+1)^2}{2(k+1) \cdot (2k+1)} = \frac{k+1}{2 \cdot (2k+1)}.\end{aligned}$$

Hence

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{k+1}{2 \cdot (2k+1)} = L = \frac{1}{2}.$$

As $L = \frac{1}{2} < 1$ we conclude that the series converges by the **Ratio test**.