Math 35: Real analysis
Winter 2018 - Homework 7

Total: 20 points | Return date: Wednesday 02/28/18

keywords: *differentiation, properties of differentiable functions*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

Chapter 4.1

exercise 7 *(2 points)* Let $f : (a,b) \to \mathbb{R}$ be a differentiable function and $c \in (a,b)$. Show that the sequence $\left(n \cdot (f(c+\frac{1}{n}) - f(c))\right)_n$ converges to $f'(c)$.

Solution: As the derivative is the limit of the function $F_c(x) = \frac{f(x)-f(c)}{x-c}$ at $x = c$ we can use the **Sequence criterion for the derivative**. The function f is differentiable at c if and only if for **any** sequence $(x_n)_n \subset (a,b)\backslash\{c\}$, such that

$$\lim_{n\to\infty} x_n = c \text{ we have } \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = L = f'(c).$$

We take the sequence where $x_n = c + \frac{1}{n}$. Then

$$\lim_{n\to\infty} c + \frac{1}{n} = c \text{ and } \lim_{n\to\infty} \frac{f\left(c+\frac{1}{n}\right) - f(c)}{\left(c+\frac{1}{n}\right) - c} = \lim_{n\to\infty} n \cdot \left(f\left(c + \frac{1}{n}\right) - f(c)\right) = L = f'(c).$$

exercise 21. *(3 points)* Find, with proof, the values of each limit.

a) $\lim_{x\to 0} \frac{\sin(x)}{x}$.
Solution: For $f(x) = \sin(x), g(x) = x$, we have $f(0) = \sin(0) = 0 = g(0)$ and $g(x) \neq 0$ in a neighborhood of $x = 0$. Hence we can apply **L'Hôpital's rule** and obtain:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\sin(x)'}{x'} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = \boxed{1}.$$

b) $\lim_{x\to 0} \frac{\cos(x)-1}{x}$
Solution: Again (check the conditions) **L'Hôpital's rule** applies and obtain:

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \lim_{x\to 0} \frac{\cos(x)'}{x'} = \lim_{x\to 0} \frac{-\sin(x)}{1} = -\sin(0) = \boxed{0}.$$

c) $\lim_{x\to 0} \frac{\sin(x)}{\tan(3x)}$
Solution: Again (check the conditions) by **L'Hôpital's rule** we obtain:

$$\lim_{x\to 0} \frac{\sin(x)}{\tan(3x)} = \lim_{x\to 0} \frac{\sin'(x)}{\tan'(3x)} = \lim_{x\to 0} \frac{\cos(x)}{3(1+\tan^2(3x))} = \frac{\cos(0)}{3(1+\tan^2(3\cdot 0))} = \boxed{\frac{1}{3}}.$$

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Chapter 4.2

exercise 15 *(2 points)* Let $[a, b]$ be an interval, such that $a > 0$. Consider the function $f : [a, b] \to \mathbb{R}, x \to f(x) = \frac{1}{x}$. Find the point c guaranteed by the **Mean value theorem**. What is special about this point?

Solution: By the **Mean value theorem** there is a point $c \in [a, b]$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We know that for $f(x) = \frac{1}{x}$ we have to find the point $c \in [a, b]$, such that

$$\boxed{-\frac{1}{c^2}} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = \left(\frac{a - b}{ab}\right) \cdot \left(\frac{1}{b - a}\right) = \boxed{-\frac{1}{ab}}.$$

Hence, as $0 < a < b$, we have that $\boxed{c = \sqrt{ab}}$.
This means that this point is the **geometric mean** of a and b.

exercise 17 *(3 points)* Use the **Mean value theorem** to show that

$$(1 + x)^{\frac{1}{2}} < 1 + \frac{x}{2} \quad \text{for all} \quad x > 0.$$

Generalize this result to the function $(1 + x)^r$, where $r > 1$.

Solution: We prove the general result for $(1 + x)^r$, where $r < 1$. The function satisfies the premises of the theorem in the interval $[0, x]$. Hence we know that there is a point $c \in [0, x]$, such that

$$\boxed{r \cdot (1 + c)^{r-1}} = f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{(1 + x)^r - (1 + 0)^r}{x} = \boxed{\frac{(1 + x)^r - 1}{x}}$$

For $r < 1$ we have that $g(c) = r \cdot (1 + c)^{r-1}$ is a decreasing function on the interval $[0, x]$. Hence the maximum is attained at $c = 0$. We obtain (as $c > 0$):

$$\boxed{\frac{(1 + x)^r - 1}{x}} = r \cdot (1 + c)^{r-1} < r \cdot (1 + 0)^{r-1} = \boxed{r} \Rightarrow (1 + x)^r < 1 + r \cdot x.$$

As this inequality does not depend on the choice of x, it is true for all $x > 0$.

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Lecture 21

Corollary 6 *(2 points)* Let $f : [a,b] \to \mathbb{R}$ be a continuous function, such that f is differentiable on (a,b), such that $m \leq f'(c) \leq M$ for all $c \in (a,b)$. Then

$$m \cdot (y-x) \leq f(y) - f(x) \leq M \cdot (y-x) \quad \text{for all} \quad x, y \in (a,b), y > x.$$

proof By the **Mean value theorem** we know that for all $x, y \in (a,b), y > x$, there is $c \in (x,y) \subset (a,b)$, such that

$$f'(c) = \frac{f(y)-f(x)}{y-x}, \quad \text{where} \quad m \leq f'(c) \leq M.$$

Hence, as $y > x \Leftrightarrow y - x > 0$ we have

$$m \leq \frac{f(y)-f(x)}{y-x} \leq M \quad \Rightarrow \quad m \cdot (y-x) \leq f(y) - f(x) \leq M \cdot (y-x).$$

This is true for all $x, y \in [a,b]$, where $y > x$.

Theorem 8 (Cauchy's mean value theorem) *(3 points)* Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions, such that f and g are differentiable on (a,b). Then there is a point $c \in (a,b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

proof We apply **Rolle's theorem** to the function

$$h(x) = (f(b)-f(a)) \cdot g(x) - (g(b)-g(a)) \cdot f(x).$$

We check whether $h(x)$ satisfies the conditions of **Rolle's theorem**. We have

$$\begin{aligned} h(a) &= (f(b)-f(a)) \cdot g(a) - (g(b)-g(a)) \cdot f(a) = f(b)g(a) - f(a)g(b) \\ h(b) &= (f(b)-f(a)) \cdot g(b) - (g(b)-g(a)) \cdot f(b) = -f(a)g(b) + f(b)g(a). \end{aligned}$$

Hence $h(a) = h(b)$ and by **Rolle's theorem** there is a point $c \in (a,b)$, such that $h'(c) = 0$. Therefore

$$\begin{aligned} 0 = h'(c) &= (f(b)-f(a)) \cdot g'(c) - (g(b)-g(a)) \cdot f'(c) \Rightarrow \\ \frac{f'(c)}{g'(c)} &= \frac{f(b)-f(a)}{g(b)-g(a)}. \end{aligned}$$

This proves our theorem.

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Chapter 4.4

exercise 2a) *(2 points)* Find the limit if f is defined on $(0, b)$ and differentiable only at c.

$$\lim_{x \to c} \frac{f(x) - f(c)}{x^2 - c^2}$$

Solution: We can apply **L'Hôpital's rule** (check the conditions) and find

$$\lim_{x \to c} \frac{f(x) - f(c)}{x^2 - c^2} = \lim_{x \to c} \frac{(f(x) - f(c))'}{(x^2 - c^2)'} = \lim_{x \to c} \frac{f'(x)}{2x} = \frac{f'(c)}{2c}.$$

Lecture 22

Exercise 10 *(3 points)* Our aim is to show that

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \text{for all} \quad x \in \mathbb{R}\,.$$

a) Show that for a fixed $x \in \mathbb{R}$ we have that

$$\lim_{y \to 0} \frac{\ln(1 + x \cdot y)}{y} = x.$$

Solution: We apply **L'Hôpital's rule** (check the conditions) and obtain

$$\lim_{y \to 0} \frac{\ln(1 + x \cdot y)}{y} = \lim_{y \to 0} \frac{\ln(1 + x \cdot y)'}{y'} = \lim_{y \to 0} \frac{x}{1 + xy} = x.$$

b) Use **Ch. 4.1, exercise 7** to show that

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Solution: This is not exactly like **Ch. 4.1, exercise 7** but very similar. By the sequence criterion for the limit we know that for **any** sequence $(y_n)_n \subset (-\delta, \delta) \backslash \{0\}$, such that

$$\lim_{n \to \infty} x_n = 0 \ \text{ we have } \ \lim_{n \to \infty} \frac{\ln(1 + x \cdot y_n)}{y_n} = x.$$

We take the sequence $y_n = \frac{1}{n}$. Then

$$\lim_{n \to \infty} \frac{1}{n} = 0 \ \text{ and } \ \lim_{n \to \infty} \frac{\ln(1 + x \cdot \frac{1}{n})}{\frac{1}{n}} = \lim_{n \to \infty} n \cdot (\ln(1 + \frac{x}{n})) = x.$$

Now $\left(n \cdot (\ln(1+\frac{x}{n})\right)_n = (x_n)_n$ is a sequence that converges to x. By the continuity of the exponential function e^x this means that $\lim_{n\to\infty} e^{x_n} = e^x$. Hence

$$\lim_{n\to\infty} e^{n\cdot\ln(1+\frac{x}{n})} = \lim_{n\to\infty} e^{\ln((1+\frac{x}{n})^n} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

This is true for all $x \in \mathbb{R}$.

Alternative solution: a,b) We realize that the limit is the derivative of $\ln(1+x\cdot y)$ at $y=0$:

$$\lim_{y\to 0}\frac{\ln(1+x\cdot y)}{y} = \lim_{y\to 0}\frac{\ln(1+x\cdot y)-\ln(1+x\cdot 0)}{y-0} = \ln(1+x\cdot y)'\big|_{y=0} = \frac{x}{1+xy}\big|_{y=0} = x.$$

Then by **Ch. 4.1, exercise 7** we have that

$$\lim_{y\to 0}\frac{\ln(1+x\cdot y)}{y} = \lim_{n\to\infty} n\cdot(\ln(1+\frac{x}{n}) - \underbrace{\ln(1+x\cdot 0)}_{=0}) = x.$$

The result then follows as in the previous solution by applying the exponential function to both sides.

c) Use part b) to show that

$$\lim_{n\to\infty}\left(1-\frac{x}{n^2}\right)^n = 1.$$

Solution: We know that (for $x>0$) we have that

$$\left(1-\frac{x}{n^2}\right) = \left(1+\frac{\sqrt{x}}{n}\right)\cdot\left(1-\frac{\sqrt{x}}{n}\right) = \left(1+\frac{\sqrt{x}}{n}\right)\cdot\left(1+\frac{(-\sqrt{x})}{n}\right).$$

Hence by the limit laws

$$\lim_{n\to\infty}\left(1-\frac{x}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{\sqrt{x}}{n}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{(-\sqrt{x})}{n}\right)^n = e^{\sqrt{x}}\cdot e^{-\sqrt{x}} = 1.$$