Math 35: Real analysis
Winter 2018 - Homework 8

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keywords: *step functions, integration*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

Lecture 23

Theorem 8 The set $T([a,b]) = \{f : [a,b] \to \mathbb{R}, f \text{ step function}\}$ is a subspace of $\mathcal{F}([a,b])$. This means we have

a) $0 \in T([a,b])$ (zero function in $T([a,b])$)

b) If $f, g \in T([a,b])$ then $f + g \in T([a,b])$.

c) If $c \in \mathbb{R}$ and $f \in T([a,b])$ then $c \cdot f \in T([a,b])$.

proof a) Clearly, for any partition P the function $O : [a,b] \to \mathbb{R}, x \mapsto O(x) = 0$ is a step function. Hence the function O is in $T([a,b])$.

b) Let $f, g \in T([a,b])$ be two step functions, such that f is a step function with partition $P_1 = \{(x_k^1)_{k=0,..,n}\}$ and g is a step function with partition $P_2 = \{(x_i^2)_{i=0,..,m}\}$.
This means that for all $k \in \{0, 1, \ldots, n-1\}$ and $i \in \{0, 1, \ldots, m-1\}$

$$f(x) = c_k \quad \text{for all} \quad x \in (x_k^1, x_{k+1}^1) \quad \text{and} \quad g(x) = d_i \quad \text{for all} \quad x \in (x_i^2, x_{i+1}^2)$$

By **Lecture 23, Lemma 7** there is a common refinement $P = \{t_0, t_1, \ldots, t_l\}$, such that $P_1 \subset P$ and $P_2 \subset P$. Then for each subinterval (t_j, t_{j+1}) we know that

$$\begin{array}{ccccc} (t_j, t_{j+1}) \subset (x_k^1, x_{k+1}^1) \text{ for some } k & \text{and} & (t_j, t_{j+1}) \subset (x_i^2, x_{i+1}^2) \text{ for some } i. \text{ Hence} \\ (f+g)(x) & = & c_k + d_i \quad \text{for all} \quad x \in (t_j, t_{j+1}). \end{array}$$

This means that $f + g$ is a step function for the partition P. Hence $f + g \in T([a,b])$.

c) Let $c \in \mathbb{R}$ be a constant. If $f \in T([a,b])$ is a step function with partition $P_1 = \{(x_k^1)_{k=0,..,n}\}$, then

$$f(x) = c_k \quad \text{for all} \quad x \in (x_k^1, x_{k+1}^1) \;\Rightarrow\; (c \cdot f)(x) = c \cdot f(x) = c \cdot c_k \quad \text{for all} \quad x \in (x_k^1, x_{k+1}^1).$$

Hence $c \cdot f$ is a step function for the partition P_1 and $c \cdot f \in T([a,b])$.

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Exercise 10 For $f, g \in T([a,b]$ show that

$$\text{a)}\quad f \cdot g \qquad \text{b)}\quad |f|^p \ \text{ for } \ p \in \mathbb{R}^+ \qquad \text{c)}\quad \min\{f,g\} \ \text{ and } \ \max\{f,g\}$$

are step functions and therefore integrable.

proof a),c) We proceed as in the proof of **Theorem 8** b):

Let $f, g \in T([a,b])$ be two step functions, such that f is a step function with partition $P_1 = \{(x_k^1)_{k=0,..,n}\}$ and g is a step function with partition $P_2 = \{(x_i^2)_{i=0,..,m}\}$.
This means that for all $k \in \{0,1,\dots,n-1\}$ and $i \in \{0,1,\dots,m-1\}$

$$f(x) = c_k \quad \text{for all} \quad x \in (x_k^1, x_{k+1}^1) \quad \text{and} \quad g(x) = d_i \quad \text{for all} \quad x \in (x_i^2, x_{i+1}^2)$$

By **Lecture 23, Lemma 7** there is a common refinement $P = \{t_0, t_1, \dots, t_l\}$, such that $P_1 \subset P$ and $P_2 \subset P$. Then for each subinterval (t_j, t_{j+1}) we know that

$$(t_j, t_{j+1}) \subset (x_k^1, x_{k+1}^1) \ \text{ for some } k \qquad \text{and} \qquad (t_j, t_{j+1}) \subset (x_i^2, x_{i+1}^2) \ \text{ for some } i.$$

As both f and g are constant on (t_j, t_{j+1}) we have for all $x \in (t_j, t_{j+1})$

$$(f \cdot g)(x) = c_k \cdot d_i \quad , \quad \min\{f(x), g(x)\} = \min\{c_k, d_i\} \quad \text{and} \quad \max\{f(x), g(x)\} = \max\{c_k, d_i\}.$$

This means that $f \cdot g, \min\{f,g\}$ and $\max\{f,g\}$ are step function for the partition P.

b) This follows in the same way as in the proof of **Theorem 8** c). If $f \in T([a,b]$ is a step function for the partition P_1 then $|f|^p$ is also a step function for the partition P_1. Hence $|f|^p \in T([a,b])$.

Chapter 5.2

exercise 19 Prove that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x = \frac{1}{n} \ \text{ for some } \ n \in \mathbb{N} \\ 0 & \phantom{\text{if}} \quad x \notin \{\frac{1}{n}, n \in \mathbb{N}\}. \end{cases}$$

is integrable on $[0,1]$.

Solution: By **Chapter 5.2, Theorem 6** it is sufficient to find a sequence of step functions $(T_n^U)_n$ and $(T_{L,n})_n$, such that

$$T_{L,n} \le f \le T_n^U \ \text{ on } [a,b] \quad \text{and} \quad \lim_{n\to\infty} \int_0^1 T_n^U - T_{L,n} = 0.$$

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The subintervals of a partition P are open in the definition of the **Darboux upper and lower sums** (see **Chapter 5.2, Def. 1**) and we take the sup and inf of f on these open intervals. By choosing a suitable partition P it is therefore sufficient that

$$T_{L,n} \leq f \leq T_n^U \quad \text{on} \quad [a,b]\backslash S \quad \text{where} \quad \#S = N \in \mathbb{N}\,.$$

Hence the inequality must be true except for a finite number of points. Therefore we can set $T_{L,n}(x) = T_L(x) = 0$ for all $x \in [0,1]$ and

$$T_n^U(x) = \begin{cases} 0 & \text{if} \quad x \in \left(\frac{1}{n}, 1\right) \\ 1 & \phantom{\text{if}} \quad x \in \left(0, \frac{1}{n}\right). \end{cases}$$

This way we obtain

$$\int_0^1 T_n^U - T_{L,n} = \int_0^1 T_n^U = \frac{1}{n}. \text{ Hence } \lim_{n\to\infty} \int_0^1 T_n^U - T_{L,n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

This means that the integral $\int_0^1 f$ exists and $\int_0^1 f = 0$.

Chapter 5.3

exercise 25 Show that the derivative of the function $f : [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x^2}\right) & \text{if} \quad x \neq 0 \\ 0 & \phantom{\text{if}} \quad x = 0. \end{cases} \qquad \text{(see below)}$$

is not integrable.

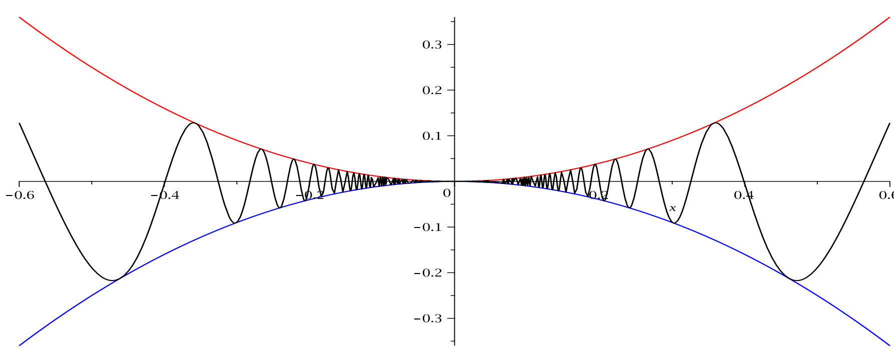

Figure 1: Plot of $f(x)$ (black) and the upper bound $g(x) = x^2$ (red) and lower bound $h(x) = -x^2$ (blue) for $x \in [-0.5, 0.5]$.

Solution: For the derivative we obtain

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) & \text{if} \quad x \neq 0 \\ 0 & \phantom{\text{if}} \quad x = 0. \end{cases} \qquad \text{(see below)}$$

Here the first part follows from the product and chain rule and the second by using the limit definition of the derivative.

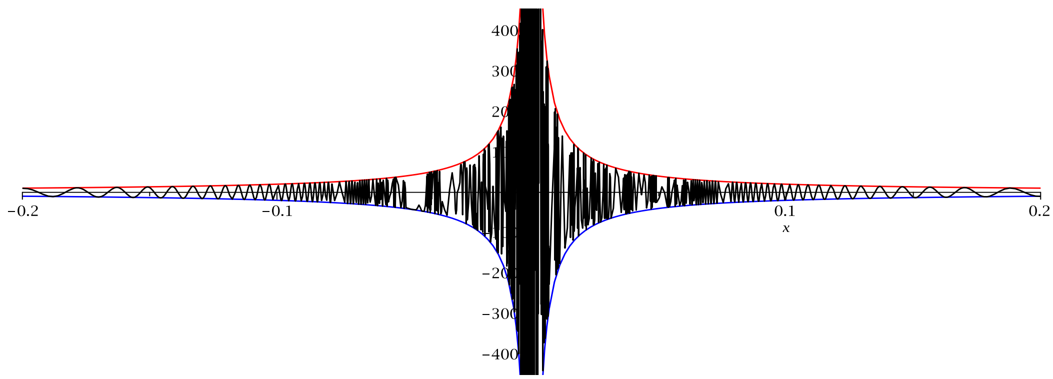

Figure 2: Plot of $f'(x)$ (black) and the upper bound $g(x) = 2(|x| + \frac{1}{|x|})$ (red) and lower bound $h(x) = -2(|x| + \frac{1}{|x|})$ (blue) for $x \in [-0.2, 0.2]$.

Consider the sequences $(x_n)_{n\in\mathbb{N}}$ where

$$x_n = \frac{1}{\sqrt{2\pi n}} \quad \text{hence} \quad f'(x_n) \quad = \quad 2x_n \underbrace{\sin(2\pi n)}_{=0} - 2\sqrt{2\pi n} \cdot \underbrace{\cos(2\pi n)}_{=1} = -2\sqrt{2\pi n}.$$

We note that

$$\lim_{n\to\infty} x_n = 0 \quad \text{but} \quad \lim_{n\to\infty} f'(x_n) = -\infty. \qquad (*)$$

Now we could argue that our function f' is unbounded on $[-1, 1]$ and the integral is only defined for bounded functions. Therefore f' is not integrable. This answer is correct.

However, we can also take a closer look and see that the lower Darboux integral does not exist:

We know that by **Lecture 25, Theorem 12** f is integrable on $[-1, 1]$ if and only if it is integrable on $[-1, 0]$ and $[0, 1]$.

Consider the interval $[0, 1]$. Let P be a partition of $[0, 1]$. Then the first subinterval defined by P is $(0, \epsilon)$ for some $\epsilon > 0$. It is easy to see that f' is bounded from above by a constant $M(\epsilon)$

on $[\epsilon, 1]$. Hence for the lower bound f_L we obtain

$$\int_0^1 f_L \leq (1-\epsilon)\cdot M(\epsilon) + \underbrace{\inf\{f(x), x \in (0,\epsilon)\}}_{=-\infty} \cdot \epsilon$$

Hence for any partition P the integral $\int_0^1 f_{L,P} = \int_0^1 f_L$ does not exist. This implies that also the lower Darboux integral $\underline{\int_0^1} f$ does not exist.

exercise 30 Use **Integration by substitution** to show that for any $b \in (0,1)$

$$\int_0^b \frac{x^3}{\sqrt{1-x^2}}\,dx = \int_0^{\arcsin(b)} \sin^3(x)\,dx = \int_{\arcsin(0)}^{\arcsin(b)} \sin^3(x)\,dx.$$

Solution: We recall the theorem:

Theorem (Integration by substitution) Let $g : [a,b] \to [c,d]$ be differentiable on $[a,b]$ and g' continuous on $[a,b]$. Let $f : [c,d] \to \mathbb{R}$ be a continuous function. Then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt.$$

The equation follows with $g(x) = \arcsin(x)$ and $f(x) = \sin^3(x)$. We have that

$$g'(x) = \arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad f(g(x)) = \sin(\arcsin(x))^3 = x^3.$$

Furthermore g is differentiable on $[0,b]$ the derivative is continuous on $[a,b]$. By the definition of $\arcsin : [-1,1) \to [-\frac{\pi}{2}, \frac{\pi}{2})$ we know that $f = \sin^3$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2})$.
Hence the conditions of the theorem are fulfilled.