Math 35: Real Analysis
Winter 2018

Friday 01/26/18

Lecture 10

Aim: A sequence converges if and only if its elements approach each other "sufficiently". This is a consequence of the completeness of $\mathbb{R}$.

Definition 3 A sequence $(a_n)_n$ is called a **Cauchy sequence** if for each $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$, such that

$$|a_n - a_m| < \epsilon \quad \text{for all} \quad n, m \geq N(\epsilon).$$

Theorem 4 Every Cauchy sequence is bounded.

proof Idea: As for **2.1.Theorem 7**. The first n values are bounded and the remaining lie in a neighborhood of the limit.
1.) Take $\epsilon = 1$. We know that there is an $N(1) = N \in \mathbb{N}$, such that

$$|a_n - a_m| < 1 \quad \text{for all} \quad n, m \geq N(1) = N.$$

In particular,

$$|a_N - a_m| < 1 \Leftrightarrow a_m \in (a_N - 1, a_N + 1) \quad \text{for all} \quad m \geq N(1) = N.$$

Hence for all $m \geq N$ we have that $|a_m| \leq \max\{|a_N - 1|, |a_N + 1|\}$.
2.) As there are only finitely many remaining elements, we know that

$$M = \max\{|a_n|, n \in \{1, 2, \ldots, N-1\} \text{ exists.}$$

In total we obtain by 1.) and 2.) that $|a_n| \leq \max\{|a_N - 1|, |a_N + 1|, M\}$ for all $n \in \mathbb{N}$. This proves our statement. □

Note: The condition that this theorem is valid for all $n, m \geq N(\epsilon)$ can not be replaced by the condition that $\lim_{n\to\infty} a_n - a_{n+1} = 0$.
The counterexample is the sequence $(a_n)_n$ where $a_n = \sum_{k=1}^n \frac{1}{k}$.

Exercise 5: a) For the above sequence $(a_n)_n$ show that $\lim_{n\to\infty} a_n - a_{n+1} = 0$.
b) Compare the sequence with $\int_1^n \frac{1}{x+1}\, dx$ to show that $\lim_{n\to\infty} a_n = \infty$.

Exercise 6: Show that the sequence $(a_n)_n$ where $a_n = \sum_{k=1}^n \frac{1}{k\cdot(k+1)}$ is a Cauchy sequence.
a) Find A, B, such that $\frac{1}{k\cdot(k+1)} = \frac{A}{k} + \frac{B}{k+1}$.
Solution: $\frac{1}{k\cdot(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

b) Use the expression from a) to find an upper bound for $|a_n - a_m|$ where without loss of generality $n > m$.
Solution:

$$|\sum_{k=1}^{n} \frac{1}{k \cdot (k+1)} - \sum_{k=1}^{m} \frac{1}{k \cdot (k+1)}| = \sum_{k=m+1}^{n} \frac{1}{k \cdot (k+1)} = \sum_{k=m+1}^{n} \frac{1}{k} - \frac{1}{k+1}$$

$$\overset{telescoping\, sum}{=} \sum_{k=m+1}^{n} \frac{1}{k} - \sum_{k=m+1}^{n} \frac{1}{k+1} = \frac{1}{m+1} - \frac{1}{n+1} \leq \frac{1}{m} \quad (*)$$

c) Conclude that $(a_n)_n$ is a Cauchy sequence.
Solution: For a given $\epsilon > 0$ we know there is $\frac{1}{N}$ such that $\frac{1}{N} < \epsilon$. Then (*) implies that

$$|a_n - a_m| < \frac{1}{N} < \epsilon \quad \text{for all} \quad m, n \geq N = N(\epsilon).$$

Hence $(a_n)_n$ is a Cauchy sequence.

Theorem 7 If $(a_n)_n$ is a converging sequence with limit a, such that

$$a_n \in [u, v] \quad \text{for all} \quad n \in \mathbb{N}\,.$$

Then $a \in [u, v]$.

proof: We only prove the inequality $a \leq v$. The inequality for the lower bound follows in the same way.
Suppose that $a > v$, then $a - v > 0$. Take $\epsilon = \frac{a-v}{2}$.

Then there is an $N = N(\epsilon) \in \mathbb{N}$, such that

$$|a_n - a| < \epsilon \Leftrightarrow a - \epsilon < a_n < a + \epsilon \quad \text{for all} \quad n \geq N.$$

Hence $v < \frac{a+v}{2} < a_n$, a contradiction. This implies that $a \leq v$. □

Theorem 8 A sequence $(a_n)_n$ is convergent if and only if it is a Cauchy sequence.

proof: " $\Leftarrow$ " Idea: As for monotone sequences the limit is an extremum.
So we know that $(a_n)_n$ is a Cauchy sequence. We define a new sequence $(b_n)_n$ where

$$b_n := \inf\{a_k, k \geq n\}$$

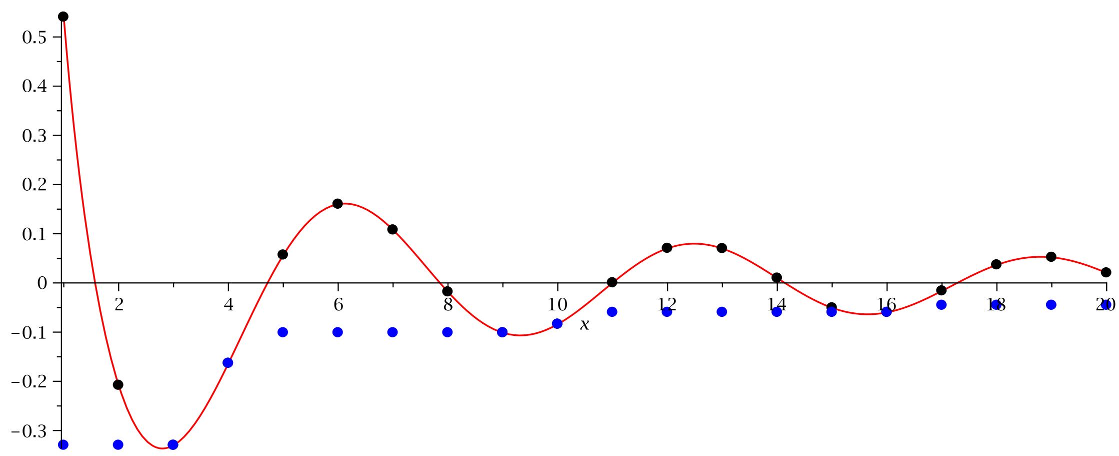

Figure 1: Plot of the sequence $(a_n)_n$, where $a_n := \frac{cos(n)}{n}$ (red) and the sequence $(b_n)_n$, where $b_n := \inf\{a_k, k \geq n\}$ (blue).

1.) We show that $(b_n)_n$ converges

We write the set of elements of $(a_n)_n$ as

$$A_1 = \{a_k, k \geq 1\} \text{ and set } A_n = \{a_k, k \geq n\}.$$

We know that the sequence $(a_n)_n$ is bounded in some interval $[u, v]$ by **Theorem 4**. Hence

$$u \leq \inf(A_1) \leq \sup(A_1) \leq v$$

As $A_n \subset A_1 = \{a_k, k \geq 1\}$ this implies that for all $n \in \mathbb{N}$

$$u \leq \inf(A_1) \leq \inf(A_n) = b_n \leq \sup(A_n) \leq \sup(A_1) \leq v.$$

Hence the sequence $(b_n)_n$ is bounded. Furthermore as $A_{n+1} \subset A_n$ we have by the definition of b_n that $b_n \leq b_{n+1}$ for all $n \in \mathbb{N}$. Hence $(b_n)_n$ is a monotone sequence. **Theorem 1** implies that it is a converging sequence. Let $b = \lim_{n\to\infty} b_n$

2.) We show that $b = \lim_{n\to\infty} a_n$

Fix $\epsilon > 0$. By the definition of the Cauchy sequence we know that there is an $N = N(\epsilon) \in \mathbb{N}$, such that

$$|a_n - a_m| < \epsilon \quad \text{for all} \quad n, m \geq N$$

In particular,

$$|a_N - a_m| < \epsilon \Leftrightarrow a_m \in (a_N - \epsilon, a_N + \epsilon) \quad \text{for all} \quad n \geq N. \quad (*)$$

By the definition of b_m that means that

$$b_m \in [a_N - \epsilon, a_N + \epsilon] \quad \text{for all} \quad m \geq N.$$

That implies that $b \in [a_N - \epsilon, a_N + \epsilon]$ by **Theorem 7**. By (*) we have that both b and the a_n for $n \geq N$ lie in the same interval around a_N. Hence

$$|a_n - b| \leq 2\epsilon < 3\epsilon \quad \text{for all} \quad n \geq N.$$

As ϵ was chosen arbitrarily this implies that $(a_n)_n$ converges to b.

" $\Rightarrow$ " To show: If $(a_n)_n$ converges then $(a_n)_n$ is a Cauchy sequence. This can be proven with the $\triangle \neq$:

Fix $\epsilon > 0$. We know that for $\frac{\epsilon}{2} > 0$ there is $N = N(\frac{\epsilon}{2}) \in \mathbb{N}$, such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \text{for all} \quad n \geq N$$

Hence for all $n, m \geq N$

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \overset{\triangle\neq}{\leq} |a_n - a| + |a - a_m| < \epsilon$$

Hence $(a_n)_n$ is a Cauchy sequence. This concludes our proof. □