Math 35: Real Analysis
Winter 2018

Wednesday 01/31/18

Lecture 12

Corollary Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^s}$, where $s \in (0, \infty)$. Then

a) $\sum_{k=1}^{\infty} \frac{1}{k^s}$ diverges for $s \in (0, 1]$.

b) $\sum_{k=1}^{\infty} \frac{1}{k^s}$ converges for $s \in \mathbb{N}, s \geq 2$.

proof:

Note: Finding the exact value of these series is not easy. Using Fourier series one can show that

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}} \text{ (Euler, 1735) } \boxed{\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}} \quad \text{and} \quad \boxed{\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}}$$

Theorem 9 (Alternating series) Let $(a_k)_k$ be a decreasing sequence, such that

$$a_k \geq 0 \quad \text{and} \quad \lim_{k \to \infty} a_k = 0.$$

Then the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cdot a_k \quad \text{converges.}$$

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Figure: Draw a dot plot of the sequences $\left(\frac{1}{k}\right)_k$ and $\left(-\frac{1}{k}\right)_k$. Then of $\left(\sum_{k=1}^{n}(-1)^{k+1}\cdot\frac{1}{k}\right)_n$. Use **Theorem 9** to show that $\sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{1}{k}$ converges. Do you know the limit?

proof of Theorem 9 Let $(S_n)_n$ be the sequence given by $S_n := \sum_{k=1}^{n}(-1)^{k+1}\cdot a_k$. We first show that $(S_{2n})_n$ and $(S_{2n+1})_n$ converge. Then we show that they have the same limit. This implies that $(S_n)_n$ converges.

1.) The sequences $(S_{2n})_n$ and $(S_{2n+1})_n$ converges
We show that these two sequences are both monotone and bounded. This means they converge by **Ch.2.2.Theorem 1**.
$(S_{2n})_n$ is increasing: We have to show that for all $n \geq 1$: $S_{2(n+1)} \geq S_{2n} \Leftrightarrow S_{2(n+1)} - S_{2n} \geq 0$. By the definition of these two sums we have

$$S_{2(n+1)} - S_{2n} = \sum_{k=2n+1}^{2n+2}(-1)^{k+1}\cdot a_k = -a_{2n+2} + a_{2n+1} \geq 0 \quad \text{as} \quad a_{2n+1} \geq a_{2n+2}.$$

The latter is true as $(a_n)_n$ is a decreasing sequence. Hence $(S_{2n})_n$ is an increasing sequence.
$(S_{2n+1})_n$ is decreasing: Similarly we find that for all $n \geq 0$:

$$S_{2(n+1)+1} - S_{2n+1} = \sum_{k=2n+2}^{2n+3}(-1)^{k+1}\cdot a_k = a_{2n+3} - a_{2n+2} \leq 0 \quad \text{as} \quad a_{2n+3} \leq a_{2n+2}.$$

Hence $(S_{2n+1})_n$ is a decreasing sequence.
It remains to show that the two sequences are bounded. To this end we note that $S_1 \geq S_2$ and

for all $n \geq 1$:

$$S_{2n+1} - S_{2n} = a_{2n+1} \geq 0 \Leftrightarrow S_{2n+1} \geq S_{2n}$$

Hence we have

This implies that

$$S_2 \leq S_{2n} \leq S_1 \quad \text{and} \quad S_2 \leq S_{2n+1} \leq S_1 \quad \text{for all} \quad n.$$

Hence the two sequences are also bounded. In total we get that $(S_{2n})_n$ and $(S_{2n+1})_n$ converge. We set $\lim_{n\to\infty} S_{2n} = S^E$ (even indices) and $\lim_{n\to\infty} S_{2n+1} = S^O$ (odd indices). It remains to show that

2.) $\lim_{n\to\infty} S_{2n} = S^E = S^O = \lim_{n\to\infty} S_{2n+1}$.
By the limit laws we have

$$S^O - S^E = \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (S_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0.$$

Hence both subsequences have the same limit $S^O = S^E = S$.
We now prove that $\lim_{n\to\infty} S_n = S$: We know by the ϵ criterion for convergence: