Math 35: Real Analysis
Winter 2018

Friday 02/02/18

Lecture 13

Outline: We give a number of tests that assert the convergence or divergence of infinite series.

Theorem 10 (Geometric series) Suppose that $a \neq 0$ then the geometric series $\sum_{k=\mathbf{0}}^{\infty} a \cdot x^k$ diverges if $|x| \geq 0$ and converges if $|x| < 1$. In the latter case we have:

$$\sum_{k=\mathbf{0}}^{\infty} a \cdot x^k = \frac{a}{1-x}.$$

proof: **1.)** $|x| \geq 1$**:** Then the sequence $(a \cdot x^k)_k$

2.) $|x| < 1$**:** Then by the formula for the geometric sum

Theorem 11 (Ratio test for series) Let $(a_k)_k$ be a sequence, such that $a_k \neq 0$ for all $k \geq K \in \mathbb{N}$. To test the series $\sum_{k=1}^{\infty} a_k$ for convergence we evaluate the limit

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L.$$

There are three possibilities:

1.) If $L < 1$ then the series converges.

2.) If $L > 1$ then the series diverges.

3.) If $L = 1$ then the test is inconclusive.

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proof 1.) If $L < 1$ then the series converges:
Idea: We prove the absolute convergence of the series by comparing it with a suitable geometric series.
As $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L$ and $L < 1$ we know that for all $\epsilon > 0$ there is $N = N(\epsilon) \geq K$, such that

$$\left|\left|\frac{a_{k+1}}{a_k}\right| - L\right| < \epsilon \Leftrightarrow L - \epsilon < \left|\frac{a_{k+1}}{a_k}\right| < L + \epsilon \quad \text{for all} \quad k \geq N.$$

Especially, by choosing a sufficiently small ϵ we can assure that there is an $L < r < 1 \in \mathbb{R}^+$, such that

$$0 < \left|\frac{a_{k+1}}{a_k}\right| < r < 1 \quad \text{for all} \quad k \geq N$$

(Take for example $\epsilon = \min\{\frac{L}{2}, \frac{1-L}{2}\}$). Hence starting at $k = N$ we know that

We conclude that

$$\begin{aligned}\sum_{k=N}^{N+n} |a_k| &= |a_N| + |a_{N+1}| + |a_{N+2}| + \ldots + |a_{N+n}| \\ &\leq |a_N| + |a_N| \cdot r + |a_N| \cdot r^2 + \ldots + |a_N| \cdot r^n = \sum_{k=0}^{n} |a_N| \cdot r^k.\end{aligned}$$

Hence in total we obtain that

$$\sum_{k=N}^{\infty} |a_k| \leq \sum_{k=0}^{\infty} |a_N| \cdot r^k = \frac{|a_N|}{1-r}.$$

Hence by the **Comparison test** we know that the series $\sum_{k=1}^{\infty} |a_k|$ converges. This means that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

2.) The series diverges for $L > 1$.
In this case it is sufficient to note that $\lim_{k\to\infty} a_k \neq 0$. Then it follows from **Theorem 6** that the series $\sum_{k=N}^{\infty} |a_k|$ can not converge.

As in Case 1.) we know that again by the ϵ-criterion of convergence we can assume that there is $R \in \mathbb{R}^+$, such that

$$\left|\frac{a_{k+1}}{a_k}\right| > R > 1 \quad \text{for all} \quad k \geq N \geq K.$$

Hence

$$|a_{k+1}| > R \cdot |a_k| > |a_k| \quad \text{for all} \quad k \geq N \geq K.$$

In a similar fashion as in part 1.) we conclude that

$$|a_k| > |a_N| \neq 0 \quad \text{for all} \quad k \geq N \geq K.$$

Hence $\lim_{k\to\infty} |a_k| \neq 0 \Leftrightarrow \lim_{k\to\infty} a_k \neq 0$ and we can conclude that $\sum_{k=N}^{\infty} a_k$ is divergent.

Examples: For which $x \in \mathbb{R}$ are the following series are convergent or divergent.

$$\text{a)} \quad \sum_{k=1}^{\infty} \frac{x^k}{k^2+k} \qquad\qquad \text{b)} \quad \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

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Theorem 12 (Root test for series) Let $(a_k)_k$ be a sequence. To test the series $\sum_{k=1}^{\infty} a_k$ for convergence we evaluate the limit

$$\lim_{k\to\infty} (|a_k|)^{\frac{1}{k}} = L.$$

There are three possibilities:

1.) If $L < 1$ then the series converges.

2.) If $L > 1$ then the series diverges.

3.) If $L = 1$ then the test is inconclusive.

proof This proof is very similar to the proof of the ratio test. Here the comparison with a suitable geometric series is even easier.

1.) If $L < 1$ then the series converges:
We know that

$$\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = L < 1.$$

Especially, by choosing a sufficiently small ϵ we can assure that there is an $r \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that

$$|a_k|^{\frac{1}{k}} < r < 1 \quad \text{for all} \quad k \geq N.$$

This implies that $|a_k| < r^k$ for all $k \geq N$. Hence

2.) If $L > 1$ then the series diverges:
As in the proof of the ratio test it is sufficient to note that $\lim_{k\to\infty} a_k \neq 0$.
As in Case 1.) we know that again by the ϵ-criterion of convergence we can assume that there is $R > 1 \in \mathbb{R}$, such that

$$|a_k|^{\frac{1}{k}} > R > 1 \Rightarrow |a_k| > R^k > 1 \quad \text{for all} \quad k \geq N \geq K.$$

Especially as R^k is increasing we have that

$$\lim_{k\to\infty} |a_k| \geq \lim_{k\to\infty} R^k = \infty.$$

Hence $\lim_{k\to\infty} a_k$ does not exist and we conclude that the series $\sum_{k=1}^{\infty} a_k$ diverges. $\square$

Example: Show that the we can easily use the root test to show that the following series converges but the ratio test is complicated.

$$\sum_{k=1}^{\infty} \left(\frac{k}{k^2+1} \right)^k$$

Solution: With the **Root test** we obtain:

$$\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = \lim_{k\to\infty} \left(\frac{k}{k^2+1} \right)^{\frac{k}{k}} = \lim_{k\to\infty} \frac{k}{k^2+1} = L = 0.$$

However with the **Ratio test** we get

$$\begin{aligned} 0 \le \frac{|a_{k+1}|}{|a_k|} &= \left(\frac{k+1}{(k+1)^2+1} \right)^{k+1} \cdot \left(\frac{k^2+1}{k} \right)^k = \left(\frac{k+1}{(k+1)^2+1} \right) \cdot \left(\frac{k+1}{(k+1)^2+1} \cdot \frac{k^2+1}{k} \right)^k \\ &= \left(\frac{k+1}{k^2+2k+2} \right) \cdot \left(\underbrace{\frac{k^3+k^2+k+1}{k^3+2k^2+2k}}_{\le 1} \right)^k \le \frac{k+1}{k^2+2k+2}. \end{aligned}$$

This is true as for $k \ge 1$

$$r = \frac{k^3+k^2+k+1}{k^3+2k^2+2k} \le 1 \Leftrightarrow k^3+k^2+k+1 \le k^3+2k^2+2k \Leftrightarrow 1 \le k^2+k.$$

Furthermore $r \le 1$ implies that $r^k \le 1$ for all $k \ge 1$.
Hence by the **Squeeze theorem** we get

$$0 \le \lim_{k\to\infty} 0 \le \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \le \lim_{k\to\infty} \frac{k+1}{k^2+2k+2} \le 0$$

Hence $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = L = 0 < 1$ and the series also converges by the **Ratio test**. However, the calculations were much easier with the root test.