Math 35: Real Analysis
Winter 2018

Wednesday 02/28/18

Lecture 24

Theorem 6 A bounded function $f : [a, b] \to \mathbb{R}$ is **integrable** if and only if for every $\epsilon > 0$ there are step functions $T^U_\epsilon = T^U, T_{L,\epsilon} = T_L \in T([a, b])$, such that

$$T_L \leq f \leq T^U \quad \text{and} \quad \int_a^b T^U(x)\,dx - \int_a^b T_L(x)\,dx \leq \epsilon.$$

Especially for the given step functions we have by the definition of the integral

$$|\int_a^b T^U(x)\,dx - \int_a^b f(x)\,dx| \leq \epsilon \;\text{ and }\; |\int_a^b f(x)\,dx - \int_a^b T_L(x)\,dx| < \epsilon.$$

proof This follows directly from the definition.

Theorem 7 (continuous functions are integrable) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is **integrable** on the interval $[a, b]$.

Figure Example for **Theorem 7**. Take an equidistant partition.

proof Idea: A continuous function on $[a, b]$ is uniformly continuous. The idea is to use **Theorem 6** and construct explicit step functions that appproximate f.

Fix $\epsilon > 0$. By **Lecture 19, Theorem 4** we know that f is uniformly continuous. Hence for the given ϵ there is $\delta(\epsilon) = \delta$, such that for all $x, \tilde{x} \in [a, b]$

$$|x - \tilde{x}| < \delta \Rightarrow |f(x) - f(\tilde{x}| < \epsilon. \quad (*)$$

We now construct our step functions:

1.) Partition P: We first choose a partition. In this case it is practical to choose an equidistant partition $P = \{(t_k)_{k=0,..,n}\}$.

$$\text{For} \quad \frac{b-a}{n} < \delta, \quad \text{set} \quad t_k = a + k \cdot \frac{b-a}{n}. \quad \text{Hence} \quad \Delta t_k = \|P\| = \frac{b-a}{n} < \delta. \quad (**)$$

2.) Upper and lower step functions f^U and f^L: We set

$$\begin{aligned} M_k &= \max\{f(x), x \in [t_k, t_{k+1}]\} \quad \text{and} \quad f^U(x) = M_k \quad \text{for all} \quad x \in (t_k, t_{k+1}) \\ m_k &= \min\{f(x), x \in [t_k, t_{k+1}]\} \quad \text{and} \quad f_L(x) = m_k \quad \text{for all} \quad x \in (t_k, t_{k+1}). \end{aligned}$$

By the **Extreme value theorem** we know that

$$M_k = f(\xi_k) \text{ and } m_k = f(\tilde{\xi}_k) \qquad \text{for some} \qquad \xi_k, \tilde{\xi}_k \in [t_k, t_{k+1}] \quad \text{hence by (*),(**)}$$
$$|\xi_k - \tilde{\xi}_k| \leq \frac{b-a}{n} < \delta \qquad \Rightarrow \qquad |f(\xi_k) - f(\tilde{\xi}_k)| = M_k - m_k < \epsilon.$$

Clearly for the step functions we have $f_L \leq f \leq f^U$. For the difference of the integrals we get

$$\int_a^b f^U(x)\,dx - \int_a^b f_L(x) = \sum_{k=0}^{n-1} M_k \cdot \Delta t_k - \sum_{k=0}^{n-1} m_k \cdot \Delta t_k = \sum_{k=0}^{n-1} \underbrace{(M_k - m_k)}_{<\epsilon(*),(**)} \cdot \underbrace{\Delta t_k}_{=\frac{b-a}{n}} < \epsilon \cdot (b-a).$$

As ϵ was chosen arbitrarily this is true for any ϵ. Hence f is integrable by **Theorem 6**. □

Theorem 8 (Linearity and monotonicity of the integral for functions)
Let $f, g : [a,b] \to \mathbb{R}$ be two integrable functions then

a) $\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$.

b) For $c \in \mathbb{R}$ we have that $\int_a^b c \cdot f(x)\ dx = c \cdot \int_a^b f(x)\ dx$.

c) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.

proof Idea: This follows from the corresponding theorem for step functions **Lecture 23, Theorem 9** and the fact that any integrable function can be "approximated" by step functions.

Example: a) By **Theorem 6** we know that for fixed $\epsilon > 0$ there exist step functions $f_\epsilon^U = f^U, f_{L,\epsilon} = f_L, g_\epsilon^U = g^U, g_{L,\epsilon} = g_L \in T([a,b])$, satisfying

$$f_L \leq f \leq f^U \quad \text{and} \quad g_L \leq g \leq g^U \quad \text{s.th.} \quad \int_a^b f^U(x) - f_L(x)\,dx < \frac{\epsilon}{2} \quad \text{and} \quad \int_a^b g^U(x) - g_L(x)\,dx < \frac{\epsilon}{2}$$

Hence

$$f_L + g_L \leq f + g \leq f^U + g^U \quad \text{and} \quad \int_a^b (f^U(x) - f_L(x)) - (g^U(x) - g_L(x))\,dx < \epsilon.$$

As ϵ was chosen arbitrarily this means that $f + g$ is integrable and part a) holds.

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Example: Calculate $\int_0^1 x^2\,dx$ using the approach from **Theorem 7** by dividing the interval into n equidistant subinterval and calculating the upper and lower step function. Recall that $\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}$:

Solution: We calculate the lower bound. For fixed n we have the partition $P = (x_k)_{k=0,..,n}$ of $[0,1]$ such that $\boxed{x_k = \dfrac{k}{n}}$. The intervals are $(\frac{k}{n}, \frac{k+1}{n})$ and we have $\boxed{\|P\| = \Delta x_k = \dfrac{1}{n}}$.
As $f(x) = x^2$ is an increasing function on the interval $[0,1]$ we know that the minimal value in each subinterval is the left endpoint. Hence for our step function $f_L = f_{L,P}$

$$f_L(x) = f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^2 \quad \text{for all} \quad x \in \left(\frac{k}{n}, \frac{k+1}{n}\right).$$

Integrating f_L we obtain:

$$\int_0^1 f_L(x)\,dx = \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\Delta x_k = \sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\cdot\sum_{k=0}^{n-1} k^2 = \frac{1}{n^3}\cdot\frac{(n-1)\cdot n\cdot(2n-1)}{6}.$$

Taking the limit $n \to \infty$ we obtain:

$$\lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{(n-1)\cdot n\cdot(2n-1)}{6} = \lim_{n\to\infty}\frac{2n^2-3n+1}{6n^2} = \frac{2}{6} = \frac{1}{3}.$$

Note: It follows from the proof of **Theorem 7** that the limit $n \to \infty$ must exist. In fact this is still true if we take any point $c_k \in (x_k, x_{k+1})$ instead of the minimum to construct a step function T such that $T(x) = f(c_k)$ on (x_k, x_{k+1}) and take finer and finer partitions.

Theorem 9 (Cauchy-Schwarz inequality for integration)
Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions. Then f^2 and g^2 are continuous functions and we have

$$|\int_a^b f(x)g(x)\,dx|^2 \leq \left(\int_a^b f^2(x)\,dx\right)\cdot\left(\int_a^b g^2(x)\,dx\right)$$

proof exercise
Hint: The proof of **Theorem 7** shows that we can find approximating lower and upper step functions for f and g on an equidistant partition.

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Theorem 10 (Mean value theorem of integration) Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions, such that $g(x) \geq 0$ for all $x \in [a, b]$. Then there is $c \in (a, b)$, such that

$$\int_a^b f(x)g(x)\,dx = f(c) \cdot \int_a^b g(x)\,dx.$$

proof We set

$$m = \min\{f(x), x \in [a, b]\} = f(u) \quad \text{and} \quad M = \max\{f(x), x \in [a, b]\} = f(v).$$

Then as $g \geq 0$ we know that $m \cdot g \leq f \cdot g \leq M \cdot g$ and by **Theorem 8** b,c) we have

$$\underbrace{m}_{=f(u)} \cdot \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \underbrace{M}_{=f(v)} \cdot \int_a^b g(x)\,dx.$$

If $\int_a^b g(x)\,dx = 0$ then it follows from this inequality that $\int_a^b f(x)g(x) = 0$ and our statement is true. If $\int_a^b g(x)\,dx \overset{g\geq 0}{>} 0$ then we can divide the inequality by this value and obtain

$$f(u) \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq f(v).$$

By the **Mean value theorem** there is a $c \in [a, b]$, such that

$$f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \Rightarrow f(c) \cdot \int_a^b g(x)\,dx = \int_a^b f(x)g(x)\,dx.$$

and again our statement is true.