Math 35: Real Analysis
Winter 2018

Tuesday 01/16/18

Lecture 5

Last time: **Theorem 10 (Cauchy-Schwarz inequality)** Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ be two vectors. Then

$$|\mathbf{a} \bullet \mathbf{b}|^2 \leq \|\mathbf{a}\|^2 \cdot \|\mathbf{b}\|^2, \qquad \text{where}$$

$$\mathbf{a} \bullet \mathbf{b} = \sum_{k=1}^{n} a_k \cdot b_k \qquad \text{and} \qquad \|\mathbf{a}\| = (\mathbf{a} \bullet \mathbf{a})^{\frac{1}{2}} = \left(\sum_{k=1}^{n} a_k^2 \right)^{\frac{1}{2}}.$$

We conclude this chapter with the following corollary:

Corollary 11: ($\triangle \neq$ in $\mathbb{R}^n$) Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ be two vectors. Then

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Figure:

proof: We prove the equivalent statement which we obtain by squaring both sides:

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \Leftrightarrow (\|\mathbf{a} + \mathbf{b}\|)^2 \leq (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.$$

We rewrite the left-hand side of the equation using the dot product. Then we use the linearity of the dot product:

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Chapter 1.5 - Completeness Axiom

Aim: $\mathbb{Q}$ is incomplete, as it misses numbers like $\sqrt{2}$ or π. We can fix this defect by adding suprema.

We start with the definition of bounded sets:

Definition 1 Let S be a non-empty set of real numbers then

a) the set S is **bounded above** if there is an $M \in \mathbb{R}$, such that

$$x \leq M \quad \text{for all} \quad x \in S.$$

In this case M is called an **upper bound** of S.

b) the set S is **bounded below** if there is an $m \in \mathbb{R}$, such that

$$m \leq x \quad \text{for all} \quad x \in S.$$

In this case m is called a **lower bound** of S.

c) the set S is **bounded** if there is an $M_a \in \mathbb{R}$, such that

$$|x| \leq M_a \quad \text{for all} \quad x \in S.$$

In this case M_a is called a **bound** of S.

Examples: - Find a set S that has an upper bound, but no lower bound.

- What can you say about the set $-S = \{-x, x \in S\}$?

- What is the greatest lower bound of the set $\tilde{S} := \{\frac{1}{n}, n \in \mathbb{N}\}$.

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Definition 2 (Supremum and Infimum) Let S be a non-empty set of real numbers.

a) If the set S is **bounded above** then a number β is the **supremum** of S or shortly $\beta = \sup(S)$ if β is an upper bound of S,i.e.

$$x \le \beta \quad \text{for all} \quad x \in S.$$

and for any number $b < \beta$ we have that b is not an upper bound of S.
This means that for all $b < \beta$ there is an $x \in S$, such that $b < x$.
The supremum is also called the **least upper bound**.

b) If the set S is **bounded below** then a number α is the **infimum** of S or shortly $\alpha = \inf(S)$ if α is a lower bound of S,i.e.

$$x \ge \alpha \quad \text{for all} \quad x \in S.$$

and for any number $a > \alpha$ we have that a is not a lower bound of S.
This means that for all $a > \alpha$ there is an $x \in S$, such that $x < a$.
The infimum is also called the **greatest lower bound**.

Example: - Find $\sup\{x \in \mathbb{Q}, x^2 < 2\}$:

We add the final axiom for the real numbers:

Completeness Axiom: Each non-empty set $S \subset \mathbb{R}$ of real numbers that is bounded above has a supremum $\sup(S)$.

A consequence is the Archimedean property of the real numbers:

Theorem 3 (Archimedean property of the real numbers) For all $a, b \in \mathbb{R}^+$ there is $n \in \mathbb{N}$, such that

$$a \cdot n > b.$$

Figure:

proof:

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We have the following lemma:

Lemma 4 The following statements are equivalent:

1. If $a, b > 0$ then there is a positive integer $n \in \mathbb{N}$, such that $na > b$.
2. The set $\mathbb{N}$ of positive integers is not bounded above.
3. For each $x \in \mathbb{R}$ there is an integer $n \in \mathbb{Z}$, such that $n \leq x < n + 1$.
4. For each $x \in \mathbb{R}^+$ there is a positive integer $n \in \mathbb{N}$, such that $\frac{1}{n} < x$.

proof: Only 1. $\Leftrightarrow$ 4.: