# Math 35: Real Analysis
# Winter 2018

Friday 01/19/18

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## Lecture 7

**Last time**:

**Definition** Let $A$ be an arbitrary set.

a) The set $A$ is **finite**, if there is a bijective map $f : A \to \{1, 2, 3, \dots, n\}$ for some $n \in \mathbb{N}$.

b) The set $A$ is **infinite**, if it is not finite.

c) The set $A$ is **countably infinite**, if there is a bijective map $f : A \to \mathbb{N}$.

d) The set $A$ is **countable**, if it is either finite or countably infinite.

e) The set $A$ is **uncountable**, if it is not countable.

**Theorem 7** A subset of a countably infinite set is countable.

**Corollary** An infinite subset of a countably infinite set is countably infinite.

**Theorem 8** A countable union of countable sets is countable.

**proof:** It is sufficient to prove the statement for a disjoint union $A = \biguplus_{i=1}^{\infty} A_i$ of countably infinte sets $A_i$. This is true as

1.) Each union of sets $\bigcup_{i=1}^{\infty} B_i$ can be decomposed into a disjoint union $\biguplus_{i=1}^{\infty} B_i'$ of sets by removing multiple occurences.

2.) Each finite set $B_i'$ can be extended to an infinite set $\tilde{B}_i$, such that $\tilde{B}_i \cap B_k' = \emptyset$ for all $k \neq i$.

3.) If $\biguplus_{i=1}^{\infty} \tilde{B}_i$ is countably infinite, then the subset $\bigcup_{i=1}^{\infty} B_i$ is countable by **Theorem 7**.

So suppose we have a disjoint union $\biguplus_{i=1}^{\infty} A_i$ of countably infinte sets $A_i$. We list all elements of $A = \biguplus_{i=1}^{\infty} A_i$:

$$
\begin{aligned}
A_1 &= \{x_{11}, x_{12}, x_{13}, \dots, x_{1n}, \dots\} \\
A_2 &= \{x_{21}, x_{22}, x_{23}, \dots, x_{2n}, \dots\} \\
&\vdots \\
A_m &= \{x_{m1}, x_{m2}, x_{m3}, \dots, x_{mn}, \dots\} \\
&\vdots
\end{aligned}
$$

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As the factorization into primes is unique, we know that the set of positive integers $S = \{2^k \cdot 3^n, n, k \in \mathbb{N}\}$ satisfies:

$$2^{k_1} \cdot 3^{n_1} = 2^{k_2} \cdot 3^{n_2} \Leftrightarrow k_1 = k_2 \text{ and } n_1 = n_2. \quad (*)$$

Hence the assignment $f : S \to \biguplus_{i=1}^{\infty} A_i$ , defined by $f(2^k \cdot 3^n) = x_{kn}$ is a well-defined map which is bijective. Hence $\biguplus_{i=1}^{\infty} A_i$ is in one-to-one correspondence with a subset of $\mathbb{N}$, which by **Theorem 7** is countable. Hence $A = \biguplus_{i=1}^{\infty} A_i$ is also countable. □

**Exercise:** List the correspondence of the elements of $S$ with $A_1$ and with $A_2$. Then show that $f : S \to A$ is indeed bijective.

**Solution:**

1.) $f$ is surjective as for each $x_{kn} \in A$ we have that $f(2^k \cdot 3^n) = x_{kn}$.

2.) $f$ is injective as if

$$f(2^{k_1} \cdot 3^{n_1}) = x_{k_1,n_1} = x_{k_2,n_2} = f(2^{k_2} \cdot 3^{n_2})$$

then, as all elements of $A$ are different, we have that

$$k_1 = k_2 \text{ and } n_1 = n_2 \Rightarrow 2^{k_1} \cdot 3^{n_1} = 2^{k_2} \cdot 3^{n_2}.$$

Hence $f$ is injective. 1.) and 2.) imply that $f$ is bijective.

3.) For $A_1$ and $A_2$ we have:

$$\begin{array}{lll} A_1 & : & f(2^1 \cdot 3^1) = x_{11}\,,\, f(2^1 \cdot 3^2) = x_{12}\,,\, f(2^1 \cdot 3^3) = x_{13}\,,\, \ldots, f(2^1 \cdot 3^n) = x_{1n}, \ldots \\ A_2 & : & f(2^2 \cdot 3^1) = x_{21}\,,\, f(2^2 \cdot 3^2) = x_{22}\,,\, f(2^2 \cdot 3^3) = x_{23}\,,\, \ldots, f(2^2 \cdot 3^n) = x_{2n}, \ldots \end{array}$$

**Theorem 9** The following two statements are true:

1.) A set $B$ that contains an infinite subset $A \subset B$ is infinite.

2.) A set $D$ that contains an uncountable subset $C \subset D$ is uncountable.

**proof:**

1.) Suppose that $B$ is finite, i.e. $\#B = n$ for some $n \in \mathbb{N}$. Then $A$ is also finite, a contradiction.

2.) Suppose that $D$ is countable, i.e. $D$ is either finite or countably infinite. If $D$ is finite then by 1.) the subset $C$ is also finite, a contradiction, as an uncountable set is infinite. If $D$ is countably infinite then by **Theorem 7** $C$ is countably infinite, again a contradiction. Hence $D$ must be uncountable.

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**Theorem 10** The set $\mathbb{Q}$ is countably infinite.

**proof:** Try to subdivide $\mathbb{Q}$ into countably infinite subsets.
**Solution:** $\mathbb{Q} = \bigcup_{i=1}^{\infty} A_i$ where

$$\begin{aligned}
A_1 &= \{\ldots, -\frac{3}{1}, -\frac{2}{1}, -\frac{1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\} \\
A_2 &= \{\ldots, -\frac{3}{2}, -\frac{2}{2}, -\frac{1}{2}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\} \\
&\vdots \\
A_n &= \{\ldots, -\frac{3}{n}, -\frac{2}{n}, -\frac{1}{n}, \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\} \\
&\vdots
\end{aligned}$$

Hence $\mathbb{Q}$ is a countable union of countably infinite sets. Then **Theorem 8** implies that it is countable. $\square$

**Theorem 11** Each real number has a binary expansion.

**proof:** HW3, look at the proof of **Theorem 1.20** in the book for inspiration.

**Theorem 12** The real numbers in the interval $[0, 1]$ are uncountable.

**proof:** By contradiction. Assume that $A = \{[0, 1]\}$ is countable. Then there is a bijective map $f : \mathbb{N} \to A$. Then we can list the elements of $A$ with their binary expansion.

**Example:** $\frac{1}{4}$ has binary expansion 0.01.
1 has the binary expansion $1.000000\ldots$, but also $0.11111111\ldots$

Choosing the latter expansion for 1 we can write each element $r \in [0, 1]$ by

$$r = 0.b_1\, b_2\, b_3 \ldots b_n \ldots, \quad \text{where} \quad b_i \in \{0, 1\} \quad \text{for all} \quad i \in \mathbb{N}\,.$$

If $A$ is countable then we can list the elements of $A$ with their binary expansion:

$$\begin{aligned}
f(1) &= 0.\boxed{b_{11}}\, b_{12}\, b_{13} \dots\ b_{1n} \dots \\
f(2) &= 0.b_{21}\, \boxed{b_{22}}\, b_{23} \dots\ b_{2n} \dots \\
f(3) &= 0.b_{31}\, b_{32}\, \boxed{b_{33}} \dots\ b_{3n} \dots \\
&\vdots \\
f(n) &= 0.b_{n1}\, b_{n2}\, b_{n3} \dots\ \boxed{b_{nn}} \dots \\
&\vdots
\end{aligned}$$

We now consider the element $x \in [0, 1]$ with binary expansion

$$x = 0.x_1\ x_2\ x_3 \dots x_n \dots, \quad \text{such that} \quad x_k = \begin{cases} 0 & \text{if} \quad b_{kk} = 1 \\ 1 & \phantom{\text{if}} \quad b_{kk} = 0. \end{cases}$$

Then $x \in [0, 1]$, but $x$ is different from all the elements in our list as it differs from each element in the list by at least on binary place. This means that the map $f : \mathbb{N} \to A$ is not surjective as it is leaving out the element $x$. Hence there is no bijection between $\mathbb{N}$ and $[0, 1]$ and the set is uncountable. $\square$

**Corollary 13**

1.) The set $\mathbb{R}$ of real numbers is uncountable.

2.) The set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable.

**proof:**

**Exercise:** Is $\mathbb{Z}^2$ countable? Is $\mathbb{Z}^3$ countable? In general is $\mathbb{Z}^n$ countable? What about $\mathbb{Z}^\infty := \{(a_k)_{k \in \mathbb{N}}, a_k \in \mathbb{Z}\}$? (Each element of $\mathbb{Z}^\infty$ can be interpreted as a vector of countably infinite length.)