Math 35: Real analysis
Winter 2018 - Midterm II

Total: *50 points* Return date: Wednesday 02/21/18

keywords: *series, limits of functions, continuity*

Instructions: Please show your work; no credit is given for solutions without work or justification. No collaboration is permitted on this exam. You may consult the textbook, your lecture notes and the homework, but no other sources (books or internet) are allowed.

problem 1 (series)

a) Suppose that both the series $\sum_{k=1}^{\infty} a_k^2$ and the series $\sum_{k=1}^{\infty} b_k^2$ converge. Show that the series $\sum_{k=1}^{\infty} a_k \cdot b_k$ converges absolutely.
Hint: Look at the lecture notes **Lecture 4**, Theorem 10.

Solution: Let $\sum_{k=1}^{\infty} a_k^2 = A$ and $\sum_{k=1}^{\infty} b_k^2 = B$. From the proof of **Lecture 4** Theorem 10 we know that

$$\sum_{k=1}^{n} |a_k \cdot b_k| \leq \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} \leq A^{\frac{1}{2}} \cdot B^{\frac{1}{2}} \quad \text{for all} \quad n.$$

Hence the sequence $(S_n)_n$ given by $S_n = \sum_{k=1}^{n} |a_k \cdot b_k|$ is monotone and bounded and therefore converges.

b) Find a series $\sum_{k=1}^{\infty} a_k$ that converges, such that $\sum_{k=1}^{\infty} a_k^2$ diverges.
Hint: Look at **Theorem 6.9** of the book.

Solution: Let $a_k = \frac{(-1)^k}{k^{\frac{1}{2}}}$. Then $a_k^2 = \frac{1}{k}$.
We know that As the sequence $\left(\frac{1}{k^{\frac{1}{2}}}\right)_k$ is decreasing and $\lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}}} = 0$. Hence by the **Alternating series test**

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\frac{1}{2}}} \quad \text{converges.}$$

However, by **HW 4** the series

$$\sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{diverges.}$$

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problem 2 If $(x_n)_n$ is a Cauchy sequence and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that $(f(x_n))_n$ is also a Cauchy sequence.

Solution: We know that $(x_n)_n$ is a Cauchy sequence. By **Lecture 10**, Theorem 8 we know that $(x_n)_n$ is a converging sequence. Hence

$$\lim_{n\to\infty} x_n = x.$$

As $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, we can apply the **Sequence criterion** of continuity. This implies that

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x).$$

Hence $(f(x_n))_n$ is a converging sequence that converges to $f(x)$. But again by **Lecture 10**, Theorem 8 we know that any converging sequence is a Cauchy sequence. Hence $(f(x_n))_n$ is a Cauchy sequence.

problem 3 Let $f : [1, +\infty) \to \mathbb{R}$ be a function, that satisfies $f(x) = f(x + 10)$. If $\lim_{x\to\infty} f(x) = 1$, show that

$$f(x) = 1 \text{ for all } x \in [1, \infty).$$

Note: Recall from **Quiz 1** that $|y| = 0 \Leftrightarrow |y| < \epsilon$ for all $\epsilon > 0$.

Solution: Let $a \in [1, \infty)$ be any point in the domain of f. As $f(x + 10) = f(x)$ for all $x \in [1, \infty)$ we know that

$$f(a + 10) = f(a)\,,\; f(a + 20) = f(a + 10) = f(a)\,,\; \ldots\,,\; f(a + 10m) = f(a) \text{ for all } m \in \mathbb{N}\,.$$

Fix $\epsilon > 0$. By the definition of the limit we know that there is $N = N(\epsilon) \in \mathbb{N}$, such that

$$|f(x) - 1| < \epsilon \quad \text{for all } x \geq N \quad (*)$$

We also know by the **Archimedean property** of the set of real numbers that there is $m \in \mathbb{N}$, such that $10m > N$, hence $a + 10m > N$. Hence by (*) we have that

$$|f(a) - 1| = |f(a + 10m) - 1| < \epsilon.$$

As ϵ can be chosen arbitrarily this implies that

$$|f(a) - 1| < \epsilon \quad \text{for all } \epsilon > 0.$$

But this means that $f(a) = 1$. As a was chosen arbitrarily we know that the statement is true for all $a \in [1, \infty)$.

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problem 4 Let $f : (-1, 1) \to \mathbb{R}$ be the function, such that

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \phantom{\text{if}} \quad x \in \mathbb{Q}\,. \end{cases}$$

a) Show that f is not continuous at any point $c \in (-1, 1)$.

Solution: Let $c \in (-1, 1)$. Then there is a sequence $(x_n)_n \subset \mathbb{Q}$ and a sequence $(y_n)_n \subset \mathbb{R} \setminus \mathbb{Q}$, such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c.$$

However by the definition of our function we have

$$\lim_{n\to\infty} f(x_n) = 0 \neq 1 = \lim_{n\to\infty} f(y_n).$$

Hence by **Lecture 16**, Theorem 2 b) f can not be continuous.

b) Find a function $g : (-1, 1) \to \mathbb{R}$ that is continuous at 0 but nowhere else.
Hint: Modify the function f.

Solution: Let $g : (-1, 1) \to \mathbb{R}$ be the function, such that

$$g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \phantom{\text{if}} \quad x \in \mathbb{Q}\,. \end{cases}$$

g is continuous for $\boxed{c = 0}$: For all $\epsilon > 0$ there is $\delta = \epsilon$, such that for all $x \in (-1, 1)$

$$|x - 0| = |x| < \epsilon \Rightarrow |g(x) - g(0)| = |g(x)| \leq |x| < \epsilon$$

Hence g is continuous at 0.
g is not continuous for $\boxed{c \neq 0}$: This follows by similar arguments as in part a).

problem 5 Let $f : [1, +\infty) \to \mathbb{R}$ and let $g : (0, 1] \to \mathbb{R}$ be the function given by $g(x) = f(\frac{1}{x})$. Show that

$$\lim_{x\to 0^+} g(x) = L \Leftrightarrow \lim_{x\to+\infty} f(x) = L.$$

Solution: "$\Rightarrow$" We know that $\lim_{x\to 0^+} g(x) = L$.
That means that for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$x \in (0, 1] \text{ and } |x - 0| = |x| < \delta \quad \Rightarrow \quad |g(x) - L| = |f\left(\frac{1}{x}\right) - L| < \epsilon. \quad (*) \quad .$$

Note that the function $h : (0,1] \to [1,+\infty), x \to h(x) = \frac{1}{x}$ is a bijective function. Hence we can set $\boxed{y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}}$ then by (*) this means that

$$|f(y) - L| < \epsilon \quad \text{for all} \quad 0 < x < \delta \Leftrightarrow y > \frac{1}{\delta}.$$

Hence for $\epsilon > 0$ we have found $N = N(\epsilon) = \frac{1}{\delta}$, such that

$$|f(y) - L| < \epsilon \ \text{ for all } \ y > N = \frac{1}{\delta}$$

This means that $\lim_{y \to +\infty} f(y) = L$.

"$\Leftarrow$" The inverse direction follows basically by reversing the steps. We state it here for the sake of completeness. We know that $\lim_{y \to +\infty} f(y) = L$.

Hence for $\epsilon > 0$ we there is $N = N(\epsilon)$, such that

$$|g(x) - L| = |f\left(\frac{1}{x}\right) - L| = |f(y) - L| < \epsilon \ \text{ for all } \ y > N \geq 1$$

Again by the bijectivity of the function $h(x) = \frac{1}{x}$, we can set $\boxed{y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}}$ then this means that

$$|g(x) - L| < \epsilon \ \text{ for all } \ y > N \Leftrightarrow 0 < x < \frac{1}{N}.$$

Hence for $\epsilon > 0$ we have found $\delta = \delta(\epsilon) = \frac{1}{N}$, such that

$$|g(x) - L| < \epsilon \ \text{ for all } \ 0 < x < \delta = \frac{1}{N}.$$

This means that $\lim_{x \to 0^+} g(x) = L$.