Math 35: Real analysis
Winter 2018 - Midterm II

Total: *50 points* Return date: Wednesday 02/21/18

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keywords: *series, limits of functions, continuity*

Instructions: Please show your work; no credit is given for solutions without work or justification. No collaboration is permitted on this exam. You may consult the textbook, your lecture notes and the homework, but no other sources (books or internet) are allowed.

problem 1 (series)

a) Suppose that both the series $\sum_{k=1}^{\infty} a_k^2$ and the series $\sum_{k=1}^{\infty} b_k^2$ converge. Show that the series $\sum_{k=1}^{\infty} a_k \cdot b_k$ converges absolutely.
Hint: Look at the lecture notes **Lecture 4**, Theorem 10.

b) Find a series $\sum_{k=1}^{\infty} a_k$ that converges, such that $\sum_{k=1}^{\infty} a_k^2$ diverges.
Hint: Look at **Theorem 6.9** of the book.

problem 2 If $(x_n)_n$ is a Cauchy sequence and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that $(f(x_n))_n$ is also a Cauchy sequence.

problem 3 Let $f : [1, +\infty) \to \mathbb{R}$ be a function, that satisfies $f(x) = f(x + 10)$. If $\lim_{x\to\infty} f(x) = 1$, show that

$$f(x) = 1 \;\text{ for all }\; x \in [1, \infty).$$

Note: Recall from **Quiz 1** that $|y| = 0 \Leftrightarrow |y| < \epsilon \;\text{ for all }\; \epsilon > 0$.

problem 4 Let $f : (-1, 1) \to \mathbb{R}$ be the function, such that

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \phantom{\text{if}} \quad x \in \mathbb{Q}\,. \end{cases}$$

a) Show that f is not continuous at any point $c \in (-1, 1)$.

b) Find a function $g : (-1, 1) \to \mathbb{R}$ that is continuous at 0 but nowhere else.
Hint: Modify the function f.

problem 5 Let $f : [1, +\infty) \to \mathbb{R}$ and let $g : (0, 1] \to \mathbb{R}$ be the function given by $g(x) = f(\frac{1}{x})$. Show that

$$\lim_{x\to 0^+} g(x) = L \Leftrightarrow \lim_{x\to +\infty} f(x) = L.$$
