ON REPRESENTATIVES OF SUBSETS

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1. Let a set S of mn things be divided into m classes of n things each in two distinct ways, (a) and (b); so that there are m (a)-classes and m (b)-classes. Then it is always possible to find a set R of m things of S which is at one and the same time a C.S.R. (= complete system of representatives) for the (a)-classes, and also a C.S.R. for the (b)-classes.

This remarkable result was originally obtained (in the form of a theorem about graphs) by D. König‡.

In the present note we are concerned with a slightly different problem, viz. with the problem of the existence of a C.D.R. (= complete system of *distinct* representatives) for a finite collection of (arbitrarily overlapping) subsets of any given set of things. The solution, Theorem 1, is very simple. From it may be deduced a general criterion, viz. Theorem 3, for the existence of a *common* C.S.R. for two distinct classifications of a given set; where it is not assumed, as in König's theorem, that all the classes have the same number of terms. König's theorem follows as an immediate corollary.

2. Given any set S and any finite system of subsets of S:

(1) $$T_1, T_2, \ldots, T_m;$$

we are concerned with the question of the existence of a *complete set of distinct representatives* for the system (1); for short, a C.D.R. of (1).

By this we mean a set of m *distinct* elements of S:

(2) $$a_1, a_2, \ldots, a_m,$$

such that

(3) $$a_i \in T_i$$

(a_i belongs to T_i) for each $i = 1, 2, \ldots, m$. We may say, a_i *represents* T_i.

It is not necessary that the sets T_i shall be finite, nor that they should be distinct from one another. Accordingly, when we speak of a system of

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† Received 23 April, 1934; read 26 April, 1934.

‡ D. König, "Uber Graphen und ihre Anwendungen", *Math. Annalen*, 77 (1916), 453. For the theorem in the form stated above, *cf.* B. L. van der Waerden, "Ein Satz über Klasseneinteilungen von endlichen Mengen", *Abhandlungen Hamburg*, 5 (1927), 185; also E. Sperner, *ibid.*, 232, for an extremely elegant proof.

k of the sets (1), it is understood that k *formally* distinct sets are meant, not necessarily k actually distinct sets.

It is obvious that, if a C.D.R. of (1) does exist, then any k of the sets (1) must contain between them at least k elements of S. For otherwise it would be impossible to find distinct representatives for those k sets.

Our main result is to show that this obviously necessary condition is also sufficient. That is

THEOREM 1. *In order that a C.D.R. of* (1) *shall exist, it is sufficient that, for each* $k = 1, 2, \ldots, m$, *any selection of* k *of the sets* (1) *shall contain between them at least* k *elements of* S.

If $A, B, \ldots$ are any subsets of S, then their *meet* (the set of all elements common to $A, B, \ldots$) will be written

$$A \wedge B \wedge \ldots.$$

Their *join* (the set of all elements which lie in at least one of $A, B, \ldots$) will be written

$$A \vee B \vee \ldots.$$

To prove Theorem 1, we need the following

LEMMA. *If* (2) *is* any *C.D.R. of* (1), *and if the meet of* all *the C.D.R. of* (1) *is the set* $R = a_1, a_2, \ldots, a_\rho$ (ρ *can be* 0, *i.e.* R *the null set*), *then the* ρ *sets*

$$T_1, T_2, \ldots, T_\rho$$

contain between them exactly ρ *elements, viz. the elements of* R.

R is, by definition, the set of all elements of S which occur as representatives of some T_i in every C.D.R. of (1).

To prove the lemma, let R' be the set of all elements a of S with the following property: there exists a sequence of suffixes

$$i, j, k, \ldots, l', l$$

such that

$$a \in T_i,$$
$$a_i \in T_j,$$
$$a_j \in T_k,$$
$$\ldots$$
$$a_{l'} \in T_l,$$

and, further,

$$l \leqslant \rho.$$

First, we shall show that every element a of R' belongs to (2). For, if not, replace, in (2),

$$a_i,\ a_j,\ a_k,\ \ldots,\ a_l$$

by

$$a,\ a_i,\ a_j,\ \ldots,\ a_{l'},$$

respectively; we obtain a new C.D.R. of (1) which does not contain a_l. Hence a_l does not belong to R, which contradicts $l \leqslant \rho$.

There will be no loss of generality in assuming that

$$R' = a_1,\ a_2,\ \ldots,\ a_\omega.$$

For it is clear that R' contains R.

Next, it is clear that if a is any element of T_i, where $i \leqslant \omega$, then

$$a \in R'.$$

For then $a_i \in R'$, and hence $j, k, \ldots, l$ can be found with $l \leqslant \rho$ and such that

$$a_i \in T_j,$$
$$\ldots$$
$$a_{l'} \in T_l.$$

And $a \in T_i$ then shows that $a \in R'$ also. Hence every element of T_i $(i \leqslant \omega)$ belongs to R'. In other words, the ω sets

$$T_1,\ T_2,\ \ldots,\ T_\omega$$

contain between them exactly ω elements, viz. the elements of R'. In *every* C.D.R. of (1), therefore, these ω T_i's are necessarily represented by these same ω elements. This shows that R' is contained in R. Hence

$$R' = R,$$

and

$$\rho = \omega,$$

and

$$R = T_1 \vee T_2 \vee \ldots \vee T_\rho.$$

This is the assertion of the lemma.

The proof of Theorem 1 now follows by induction over m. The case $m = 1$ is trivial.

We assume then that any k of the sets (1) contain between them at least k elements of S, and also that the theorem is true for $m-1$ sets. We may therefore apply the theorem to the $m-1$ sets

$$T_1,\ T_2,\ \ldots,\ T_{m-1}. \tag{4}$$

These have, accordingly, at least one C.D.R. Hence (1) will also have at least one C.D.R., provided only that T_m is not contained in *all* the C.D.R. of (4).

But if (without loss of generality)

$$R^{*} = a_1, a_2, \dots, a_\rho \quad (\rho \geqslant 0)$$

is the meet of *all* the C.D.R. of (4), and if T_m is contained in R^{*}, then, by the lemma, the $k = \rho + 1$ sets

$$T_1, T_2, \dots, T_\rho, T_m$$

contain between them only ρ elements, viz. those of R^{*}. This being contrary to hypothesis, T_m is not contained in R^{*}; and so, if a_m is any element of T_m not in R^{*}, there exists a C.D.R. of (4) in which a_m does not occur. This C.D.R. of (4) together with a_m constitutes the desired C.D.R. of (1).

An elementary transformation of Theorem 1 gives

THEOREM 2. *If S is divided into any number of classes (e.g. by means of some equivalence relation),*

$$S = S_1 \vee S_2 \vee S_3 \vee \dots,$$

and $S_i \wedge S_j$ is the null set, for $i \neq j$, then there always exists a set of m elements

$$a_1, a_2, \dots, a_m,$$

no two of which belong to the same class, such that

$$a_i \in T_i \quad (i = 1, 2, \dots, m),$$

provided only that, for each $k = 1, 2, \dots, m$, any k of the sets T_i contain between them elements from at least k classes.

Proof. Denote by t_i the set of all classes S_j for which the meet

$$S_j \wedge T_i$$

is not null. The condition to be satisfied by the sets T_i may then be expressed thus: any k of the t_i's contain between them at least k members. Applying Theorem 1, it follows that there exists a set of *m distinct* classes, for simplicity

$$S_1, S_2, \dots, S_m,$$

such that, for $i = 1, 2, \dots, m$, the set

$$S_i \wedge T_i = M$$

is not null. Choosing for a_i an arbitrary element from M_i, the result follows.

A particular case of some interest is

THEOREM 3. *If the set S is divided into m classes in two different ways,*

$$S = S_1 \vee S_2 \vee \dots \vee S_m,$$
$$S = S_1' \vee S_2' \vee \dots \vee S_m',$$

$S_i \wedge S_j = S_i' \wedge S_j' =$ *null set, for* $i \neq j$, *then, provided that, for each* $k = 1, 2, \ldots, m$, *any* k *of the classes* S_j' *always contain between them elements from at least* k *of the classes* S_i, *it will always be possible to find* m *elements of* S,

$$a_1, a_2, \ldots, a_m,$$

such that (possibly after permuting the suffixes of the S_j'*)*

$$a_i \in S_i \wedge S_i' \quad (i = 1, 2, \ldots, m).$$

The case in which all the classes have the same (finite) number of elements clearly fulfils the proviso. Theorem 3 then becomes the well-known theorem of König, referred to above.

The generalization of König's theorem due to R. Rado† may also be deduced without difficulty from Theorem 3.

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ON THE ADDITION OF RESIDUE CLASSES

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1. The object of this note is to give a proof of the following simple theorem:

A. *Let* p *be a prime; let* $\alpha_1, \ldots, \alpha_m$ *be* m *different residue classes* mod p; *let* $\beta_1, \ldots, \beta_n$ *be* n *different residue classes* mod p. *Let* $\gamma_1, \ldots, \gamma_l$ *be all those different residue classes which are representable as*

$$\alpha_i + \beta_j \quad (1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n).$$

Then $$l \geqslant m+n-1,$$

provided that $m+n-1 \leqslant p$, *and otherwise* $l = p$.

This may be described as the "mod p analogue" of a conjecture concerning the density of the sum of two sequences which is naturally suggested by the recent work of Khintchine§. I am indebted to Dr. Heilbronn for suggesting this question to me and for simplifying the method of presentation of the proof.

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† R. Rado, "Bemerkungen zur Kombinatorik im Anschluss an Untersuchungen von Herrn D. König", *Berliner Sitzungsberichte*, 32 (1933), 60, Satz I, 61.

‡ Received 16 April, 1934; read 26 April, 1934.

§ A. Khintchine, "Zur additiven Zahlentheorie", *Rec. Soc. Math. Moscou*, 39 (1932), 3, 27–34.