**Inverse Trigonometric Functions**
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We will introduce inverse functions for the sine, cosine, and tangent. In defining them, we will point out the issues that must be considered in defining the inverse of any periodic function. Then we will go on to find the derivative of the inverse sine and the inverse tangent. Their companion integration formulas will give us two new integrals that we will subsequently recognize how to calculate.
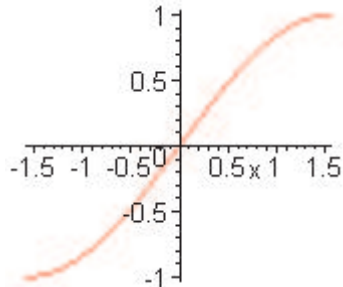
The Arcsine

We would like to define the inverse function of the sine. However, because the sine is periodic, it is not one-to-one and the graph of the sine function fails the horizontal line test. Hence the sine does not have an inverse unless we restrict its domain.
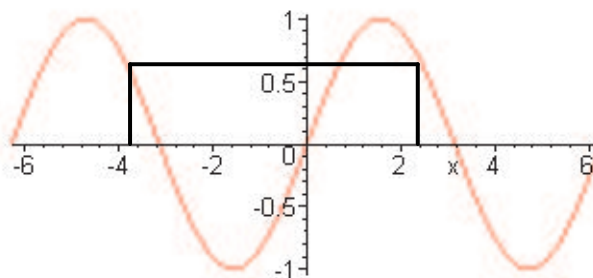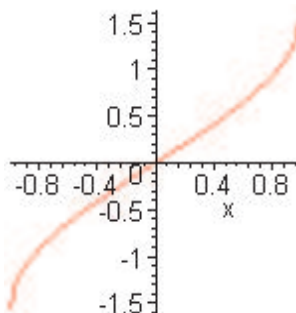
sine function
is not one-to-one

By convention we restrict the domain of the sine to the interval $[-\pi/2, \pi/2]$ where it is one-to-one of course. And we call its inverse on this restricted domain the *arcsine* function or the *inverse sine* function.

sine on restricted domain

Here is a graph of $y = \arcsin x$.

arcsine (inverse sine) function

We will formalize in a definition what we have just described.
Definition 1: Let $-1 \leq x \leq 1$. Then $y = \arcsin x$ if and only if $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$.

That is, we read $y = \arcsin x$ as: *y is the angle (in radians) between $-\pi/2$ and $\pi/2$ whose sine is equal to x.*

Example 1: Here are some values of the arcsine function:

1. $\arcsin 0 = 0$

2. $\arcsin 1 = \pi/2$

3. $\arcsin(-1) = -\pi/2$

4. $\arcsin(-1/\sqrt{2}) = -\pi/4$

5. $\arcsin(1/2) = \pi/6$

6. $\arcsin 5$ is not defined because 5 is not in the range of the sine.

Example 2: We want to keep in mind that the sine and arcsine functions have an inverse function relationship but on a restricted domain:

1. $\arcsin(\sin(\pi/3)) = \pi/3$

2. $\arcsin(\sin(3\pi/4)) = \pi/4$. Note that the answer is not $3\pi/4$ because $3\pi/4$ is outside the domain $[-\pi/2, \pi/2]$ to which we restricted the sine.

Notation: We have been writing the inverse sine function as $y = \arcsin x$. There is an alternative notation that can be used interchangeably: $y = \sin^{-1} x$. Just be careful not to interpret this to mean the reciprocal of the sine.
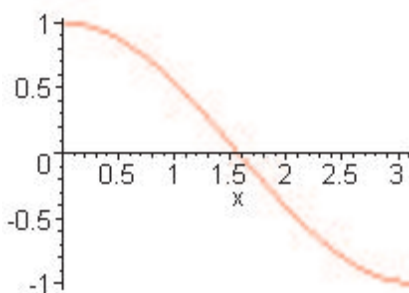
The Arccosine

Like the sine, the cosine function also fails to be invertible unless we restrict its domain. Corresponding to each value of y in the range of the cosine is an infinite number of x-values. We show part of the graph below.
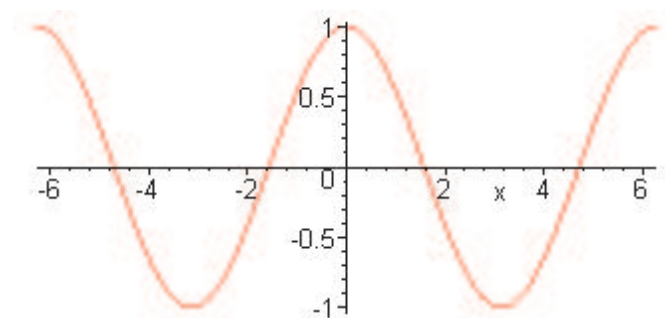

cosine function is not one-to-one

As before, we need to restrict the domain to an interval where the function is one-to-one. In the case of the cosine, the agreed upon convention is to restrict the domain to the interval $[0, \pi]$.

cosine on restricted domain

Now, we can define the inverse function by swapping domain and range and reversing the action of the cosine. That is, if the cosine maps x to y, then the arccosine maps y to x.

arccosine function
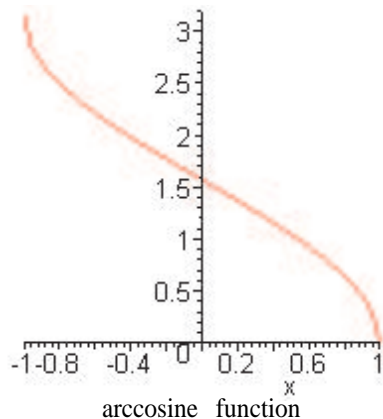
Let's summarize what we have done by collecting the information in a definition.

Definition 2: Let $-1 \leq x \leq 1$. Then $y = \arccos x$ if and only if $\cos y = x$ and $0 \leq y \leq \pi$.
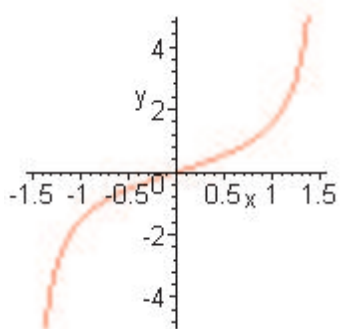
We read $y = \arccos x$ as: *y is the angle (in radians) between 0 and π whose cosine is equal to x.* As in the case of the sine, instead of $y = \arccos x$ we can just as well write $y = \cos^{-1} x$.

**Example 3:** Some values of the inverse cosine are:

1. $\arccos 1 = 0$

2. $\arccos(-1) = \pi$

3. $\arccos 0 = \pi/2$

4. $\arccos(-1/2) = 2\pi/3$

Check them for yourself, remembering the way in which we restricted the domain of the cosine.

The Arctangent

Even though the tangent function is not one-to-one on its domain, it is one-to-one on the branch that passes through the origin. By convention, we use this branch to define the inverse. That is, to define the inverse function for the tangent, we restrict the domain to the open interval $(-\pi/2, \pi/2)$. Then the tangent is one-to-one and we can define the arctangent function accordingly.
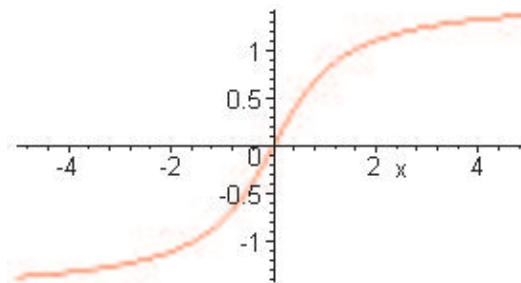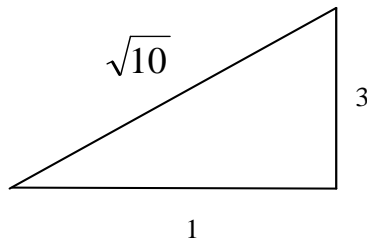
inverse tangent

Definition 3: Let $-\infty < x < \infty$. Then $y = \arctan x$ if and only if $\tan y = x$ and $-\pi/2 < y < \pi/2$.

So, according to the definition, we read $y = \arctan x$ as: *y is the angle (in radians) strictly between $-\pi/2$ and $\pi/2$ whose tangent is equal to x.*

Example 4: Here are a few values of the arctangent:

1. $\arctan 0 = 0$

2. $\arctan 1 = \pi/4$

3. $\arctan(-1) = -\pi/4$

4. $\arctan(\tan(3\pi/4)) = \arctan(-1) = -\pi/4$

Example 5: To find $\cos(\arctan 3)$, we make a triangle from the information $y = \arctan 3$ and use the Pythagorean Theorem to complete it. Thus, we see that the cosine of the angle (and hence the answer to the problem) is $1/\sqrt{10}$.

**Derivative of the Arcsine and the Arctangent**

Arcsine: Now that we have defined inverse functions for some of the trigonometric functions, we will find their derivatives. In particular, we will discover some new antiderivatives that come up frequently in integration problems.

Theorem 1: Let $y = \arcsin x$. Then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

The proof starts with the defining relationship between the sine and arcsine functions: $y = \arcsin x \Leftrightarrow \sin y = x$ and $-\pi/2 \leq y \leq \pi/2$. Now, beginning with the right hand side and differentiating implicitly, we get:

$$\sin y = x$$
$$\cos y \, \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now, make a triangle using the relationship $\sin y = x$ to determine two of the sides, and apply the Pythagorean Theorem to find the third.
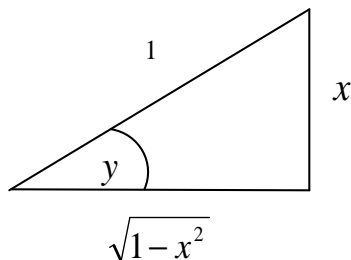
Thus, $\cos y = \sqrt{1 - x^2}$. [It may appear that we are assuming that y is a first-quadrant angle. But because the cosine is positive in the first and fourth quadrants (that is, for $-\pi/2 \le y \le \pi/2$), this expression for the cosine is indeed correct for all values of y under consideration.] Hence, we have the desired result, namely, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

As usual, this theorem has a chain-rule form, and a companion integral formula. For, if u is a function of x, then

$$\frac{d}{dx} \arcsin u = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}$$

$$\int \frac{1}{\sqrt{1 - u^2}} \, du = \arcsin u + C$$

Example 6: If $y = x \arcsin x$, then from the product rule we have $y' = \arcsin x + \frac{x}{\sqrt{1-x^2}}$.

Example 7: If $y = e^{\arcsin x}$, then $y' = e^{\arcsin x} \frac{1}{\sqrt{1-x^2}}$.

Example 8: Find $\int \frac{1}{\sqrt{a^2-x^2}} \, dx$, where a is a constant, by calculating the derivative of $\arcsin \frac{x}{a}$. Following the instructions and using the chain rule, we get:

$$\begin{aligned}
\frac{d}{dx} \arcsin \frac{x}{a} &= \frac{1}{\sqrt{1 - (x/a)^2}} \frac{1}{a} \\
&= \frac{a}{\sqrt{a^2 - x^2}} \frac{1}{a} \\
&= \frac{1}{\sqrt{a^2 - x^2}}
\end{aligned}$$

Therefore, we can solve the integral given in the Example:

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + C$$

Example 9: Find $\int \frac{1}{\sqrt{3-x^2}} \, dx$. From the previous Example, the answer is

$$\int \frac{1}{\sqrt{3 - x^2}} \, dx = \arcsin \frac{x}{\sqrt{3}} + C$$

Example 10: We can use parts to solve $\int \arcsin x \, dx$.

$u = \arcsin x$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}} \, dx$	$v = x$

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} \, dx$$

The new integral can be solved by substitution with $u = 1 - x^2$, and hence $du = -2x\,dx$.

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = -(1-x^2)^{1/2} + C$$

Thus,

$$\int \arcsin x\,dx = x\arcsin x + (1-x^2)^{1/2} + C$$

Arccosine: The derivative of the arccosine does not help us deal with integrals because we can use the arcsine in integration problems:

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

Instead of proving that result, we will go on to a proof of the derivative of the arctangent function. In spirit, all of these proofs are the same.

Arctangent: The arctangent function is defined through the relationship $y = \arctan x \Leftrightarrow \tan y = x$ and $-\pi/2 < y < \pi/2$. As we did in proving the derivative of arcsine, we will begin with the right hand side and differentiate implicitly.

Theorem 2: Let $y = \arctan x$. Then $\frac{dy}{dx} = \frac{1}{1+x^2}$.

The outline of the proof is the same as that for the derivative of the arcsine.

$$\tan y = x$$
$$\sec^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

We now draw a triangle using the relationship $\tan y = x$ to determine two sides, and use the Pythagorean Theorem for the third.
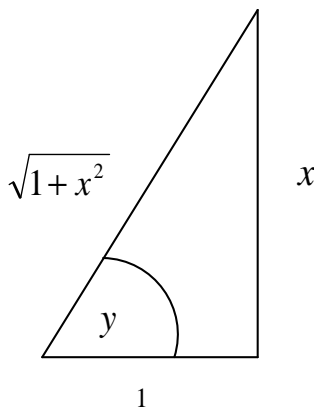
Thus, $\sec y = \frac{1}{\cos y} = \sqrt{1+x^2}$. Hence $\frac{dy}{dx} = \frac{1}{1+x^2}$, and the proof is complete.

The chain-rule form of the theorem and the companion integral formula follow: If u is a function of x, then

$$\frac{d}{dx}\arctan u = \frac{1}{1+u^2}\frac{du}{dx}$$
$$\int \frac{1}{1+u^2}\,du = \arctan u + C$$

Example 11: If $y = \arctan x^2$, then by the chain rule we have $y' = \frac{1}{1+x^4} 2x$.

Example 12: If $y = \arctan e^x$, then $y' = \frac{1}{1+e^{2x}} e^x$

Example 13: If a is constant, then the chain rule yields:

$$\frac{d}{dx} \arctan \frac{x}{a} = \frac{1}{1 + (\frac{x}{a})^2} \frac{1}{a} = \frac{a}{a^2 + x^2}$$

Thus

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \, \arctan \frac{x}{a} + C$$

Example 14: From the previous Example, we get that $\int \frac{1}{5+x^2} \, dx = \frac{1}{\sqrt{5}} \arctan \frac{x}{\sqrt{5}} + C$.

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