Exponential Growth and Decay Problems

February 11, 2014

(1) Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours? # cells = P

$$\frac{dP}{dt} = kP$$ where k is a constant

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln P = kt + C$$

$$\underbrace{P}_{P(t)} = e^{kt+C} = e^{kt}e^{C}$$

$$P(0) = 700 = e^{C}$$

$$P(t) = 700e^{kt}$$

$$P(12) = 700e^{k(12)} = 900$$

$$e^{k(12)} = 9/7$$

$$k \cdot 12 = \ln(9/7)$$

$$k = \frac{\ln(9/7)}{12}$$

$$P(t) = 700\,e^{\frac{\ln(9/7)}{12} \cdot t}$$

$$P(24) = 700e^{(\ln(9/7)/12) \cdot 24}$$

$$= \boxed{1157 \text{ cells}}$$

Simplify:

$$700e^{(\ln(9/7)/12)t}$$

$$= 700\,e^{\ln(9/7^{t/12})}$$

$$= \boxed{700\,(9/7)^{t/12}}$$

(2) Objects heat or cool at a rate proportional to the difference between their temperature and the ambient temperature of the room. Suppose a pie is pulled out of the oven (heated to 370°F), and put into a room that's 70°F. After 10 minutes, the center of the pie is 340°F. y = temperature, t = time

$$\frac{dy}{dt} = k(y-70)$$

$$\int \frac{1}{y-70}\,dy = \int k\,dt$$

$$\ln(y-70) = kt + C$$

$$y - 70 = e^{kt+C} \quad (A = e^{C})$$

$$y - 70 = Ae^{kt}$$

$$y = Ae^{kt} + 70$$

when $t=0, y=370$
when $t=10, y=340$

$$370 = Ae^{k0} + 70$$

$$370 = A + 70 \Rightarrow A = 300$$

$$340 = 300e^{k(10)} + 70$$

$$\frac{270}{300} = e^{k(10)}$$

$$\ln(9/10) = k(10)$$

$$\frac{\ln(9/10)}{10} = k$$

$$y = 300e^{\frac{\ln(9/10)}{10} \cdot t} + 70$$

$$y = 300\left(\frac{9}{10}\right)^{(t/10)} + 70$$

(a) How hot is the pie after 20 minutes?

$$300\left(\frac{9}{10}\right)^{20/10} = 300\left(\frac{9}{10}\right)^{2} + 70$$

$$= 3 \cdot 81 + 70 = 313°F$$

(b) How long will it take for the center of the pie to cool to 100°F?

$$100 = 300\left(\frac{9}{10}\right)^{t/10} + 70$$

$$\frac{30}{300} = (9/10)^{t/10}$$

$$\frac{1}{10} = (9/10)^{t/10}$$

$$\frac{\ln(1/10)}{\ln(9/10)} = t/10 \Longrightarrow 10 \cdot \frac{\ln(1/10)}{\ln(9/10)} \approx \underbrace{218.54 = t}_{\text{after } 218.54 \text{ minutes}}$$

(3) The isotope thorium-239 decays at a rate proportional to the amount present, and has a half-life of 24.1 days. How long does 10 grams of thorium-234 take to decay to 1 gram? ("Half-life": The time it takes for an amount of stuff to halve in size.)

$\frac{dP}{dt} = kP$ P = grams of thorium

$\int \frac{dP}{P} = \int k\,dt$

$\ln P = kt + C$

$P = Ae^{kt}$

at $t=0$, $P=10$

at $t=24.1$, $P=5$ (half-life)

$Ae^{k\cdot 0} = 10 \implies A = 10$

$10e^{k\cdot 24.1} = 5$

$e^{k\cdot 24.1} = \frac{1}{2}$

$k\cdot 24.1 = \ln(1/2)$

$k = \frac{\ln(1/2)}{24.1} = \ln(1/2)^{1/24.1}$

so $P = 10e^{\ln(1/2)^{1/24.1}\cdot t}$

$= 10e^{\ln(1/2)^{t/24.1}}$

$= 10\cdot\left(\frac{1}{2}\right)^{t/24.1}$

Want time when P = 1 gram.

$1 = 10\left(\frac{1}{2}\right)^{t/24.1}$

$\frac{1}{10} = \left(\frac{1}{2}\right)^{t/24.1}$

$\log_{1/2}(1/10) = t/24.1$

$$\underbrace{\frac{\ln(1/10)}{\ln(1/2)}\cdot 24.1}_{80.06 \text{ days}} = t$$