Math 40 Probability and Statistical Inference

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Yoonsang Lee (yoonsang.lee@dartmouth.edu)

Examples of continuous random variables

Uniform (section 2.3)

- X$\sim \mathcal{R}(a, b)$ or $Uniform(a, b)$.
- pdf $f(x) = \frac{1}{b-a}$, $x \in (a, b)$.
- cdf $F(x) = \frac{x-a}{b-a}$, $x \in (a, b)$.

Exponential (section 2.4)

- Waiting time or survival analysis
- X$\sim \mathcal{E}(\lambda)$, λ: rate.
- pdf $f(x;\lambda) = \lambda e^{-\lambda x}$.
- cdf $F(x;\lambda) = 1 - e^{-\lambda x}$.
- $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda}^2$.
- An interpretation of $F(x)$:
  probability of death by time x.
- $S(x) = 1 - F(x)$ survival function:
  probability of survival until x.
- $H(x) = \frac{-S'(x)}{S(x)}$ hazard function:
  instantaneous relative mortality

Exponential (section 2.4)

Example 2.16 Waiting for a call from Bill starting from 10 am. If the call follows an exponential distribution with $\lambda = 1/10$,
(a) Probability of having a call between 10:00 am to 10:10 am
(b) Probability of having a call between 10:10 am to 11:00 am.

Laplace (or double exponential) (section 2.4.1)

- $X \sim \mathcal{L}(\lambda)$.
- pdf $f(x;\lambda) = \frac{\lambda}{2} e^{-\lambda|x|}$
- cdf $F(x;\lambda) = \begin{cases} \frac{\lambda}{2} e^{\lambda x}, & \text{if } x<0 \\ 1 - \frac{\lambda}{2} e^{\lambda x} & \text{if } x \geq 0 \end{cases}$
- has many applications in sparse reconstruction.

Gamma distribution (section 2.6)

- $X \sim \mathcal{G}(\alpha, \lambda)$, α: shape, λ: rate.
- When $\alpha \in \mathbb{N}$, X is the sum of α independent exponential distributions of the same rate λ.
- $f(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $x > 0$
- $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$, $\alpha > 0$ satisfying
  1. $\Gamma(1) = \Gamma(2) = 1$
  2. $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$
  3. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$
  4. $\Gamma(0.5) = \sqrt{\pi}$
- $E(X) = \frac{\alpha}{\lambda}$, $Var(X) = \frac{\alpha}{\lambda^2}$

Relationship between Poisson and Gamma

$X \sim Poisson(\lambda)$ and $Y \sim \Gamma(\alpha, \lambda)$

$$F_{poisson}(\alpha - 1; \lambda) = 1 - F_{gamma}(1; \alpha, \lambda)$$

Beta distributions (section 2.14)

- $X \sim \mathcal{B}(\alpha, \beta)$, α, β : shape
- pdf $f(x; \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0$
- Here $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- $E(X) = \frac{\alpha}{\alpha+\beta}$, $Var(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
- Relationship between Beta and Binomial

$$\int_0^p f(x; k, n-k+1)dx = 1 - \sum_{m=0}^{k-1} \binom{n}{m} p^m (1-p)^{n-m}$$