Math 40 Probability and Statistical Inference

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Normal and lognormal distributions

Normal distributions (section 2.7)

- $X \sim \mathcal{N}(\mu, \sigma^2)$
- pdf $\phi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$
- $E(X) = \mu,\ Var(X) = \sigma^2$
- Standard normal if $\mu = 0$ and $\sigma^2 = 1$
- Standard normal pdf $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
- Standard normal cdf $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$
- cdf of $\mathcal{N}(\mu, \sigma^2) = \Phi(\frac{x-\mu}{\sigma})$.
- MGF of the standard normal (Example 2.31)

$$M(t) = e^{\frac{t^2}{2}}$$

Normal distributions (section 2.7)

Example 2.32 The rule of two sigma. Compute the probability $Pr(|X - \mu| < 2\sigma)$, where $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 1$ and $\sigma = \sqrt{2}$.

Normal distributions (section 2.7)

Example 2.34 Volume of the ball. The radius of a ball is $\rho = 2.5$ cm, but it is measured with error $\sigma = 0.001$ cm. Assuming that the measurement is normally distributed, find the probability that the volume of the ball

1. is greater than the nominal volume $(4/3)\pi\rho^3 = 65.45\text{cm}^3$ and
2. is within ± 0.01 of the nominal volume.
3. Is an estimator of the volume, $(4/3)\pi\bar{r}^3$, where $\bar{r}$ is an unbiased normally distributed measurement of the radius, unbiased?

Normal distributions (section 2.7)

Example 2.32 The rule of two sigma. Compute the probability $Pr(|X - \mu| < 2\sigma)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 1$ and $\sigma = \sqrt{2}$ using the **pnorm** function.

Normal distributions (section 2.7)

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Normal distributions (section 2.7)

Theorem 2.35 The expected value of the normal cdf can be expressed in closed form as

$$E_{X \sim \mathcal{N}(\mu,\sigma^2)} \Phi(X) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$$

Normal distributions (section 2.7)

Definition 2.37 A continuous distribution on $(-\infty, \infty)$ is called skew-normal if its density is defined as

$$f(x; \mu, \sigma^2, \lambda) = \frac{2}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\lambda\frac{x-\mu}{\sigma}\right)$$

We will use this distribution in statistics to model a skewed distribution.

Exercise 2.7.4

Prove that Y^2 has a gamma distribution if $Y \sim \mathcal{N}(0, 1)$. Find α and λ.

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Prove that Y^2 has a gamma distribution if $Y \sim \mathcal{N}(0,1)$. Find α and λ. pdf $f(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}$ pdf of $gamma(\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$. Note $\Gamma(1/2) = \sqrt{\pi}$

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Exercise 2.7.6

Prove that for a fixed $\tau > 0$, $Pr(|X - a| < \tau)$ takes maximum when $a = \mu$ where $X \sim \mathcal{N}(\mu, \sigma^2)$.

Lognormal distributions (section 2.11)

- $X = e^U$ where $U \sim \mathcal{N}(\mu, \sigma^2)$
- pdf $f(x; \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln x - \mu)^2}, x > 0$
- cdf $F(x; \mu, \sigma^2) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$