Math 40 Probability and Statistical Inference

Winter 2021

Yoonsang Lee (yoonsang.lee@dartmouth.edu)

Lecture 6: Law of large numbers and central limit theorem

Chebyshev's inequality (section 2.8)

- **Markov's inequality** For a positive random variable Z

$$Pr(Z > t) \leq \frac{E(Z)}{t}$$

- **Chebyshev's inequality** For a random variable with mean μ and variance σ^2,

$$Pr(|X - \mu| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}$$

Chebyshev's inequality (section 2.8)

Exercise Will you consider a coin asymmetric if after 1000 coin tosses the number of heads is equal to 600?

Law of large numbers (section 2.9)

Theorem 2.43 The law of large numbers (LLN)
(Weak) If $X_1, X_2, \dots$ are **uncorrelated** random variables with the same mean $E(X_i) = \mu$ and the variance $Var(X_i) = \sigma^2$, then $\overline{X}_n = \frac{1}{n}\sum_i^n X_i$ converges (in probability) to μ.
(Strong) If $X_1, X_2, \dots$ are iid (independent and identically distributed) and have finite mean μ, then $\overline{X}_n$ converges to μ almost surely.

Law of large numbers (section 2.9)

Example Let X_i be the outcome of Bernoulli(p) at the i-th trial.
We estimate p using $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n$
From LLN, we know that $\overline{X} \to \mu$ as $n \to \infty$.

Law of large numbers (section 2.9)

Example 2.44 Consistency of the ruler

- X: measurement of the ruler, two outcomes r or $r+1$ with probability $(r+1)-L$ and $L-r$ respectively.
- The expected value $E(X)$ is L.
- So, from LLN, if you take average of several measurements, X_i,i=1,2,..., the average converges to L as $n \to \infty$.

Law of large numbers (section 2.9)

Example 2.48 Consistency of the sample mean, variance, and SD For a IID sample $\{X_i\}$ from a distribution with mean μ and variance σ^2,

- Sample mean $\hat{\mu} = \overline{X}_n = \frac{1}{n}\sum_i^n X_i$
- Sample variance $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i^n}(X_i - \overline{X}_n)^2$

These estimators are consistent, that is, they converges to the true values μ and σ^2 respectively as $n \to \infty$.

Law of large numbers (section 2.9)

Integral approximation using simulations (Monte-Carlo integration

$$\int_a^b f(x)dx = (b-a)\int_a^b f(x)\frac{1}{b-a}dx = (b-a)E_{X\sim\mathcal{R}(a,b)}(f(X))$$

From LLN, we can approximate the integral

$$\int_a^b f(x)dx = (b-a)\frac{1}{n}\sum_i^n f(x_i)$$

where $\{x_i\}$ is a sample from $\mathcal{R}(a,b)$.

Law of large numbers (section 2.9)

Integral approximation using simulations (Monte-Carlo integration

Q: What if your integral domain is semi-infinite

$$\int_0^{\infty} f(x)dx$$

A: Use the following change of variables

$$\int_0^{\infty} f(x)dx = \int_0^{\infty} f(x)\frac{e^{\lambda x}}{\lambda}\lambda e^{-\lambda x}dx$$

$$\approx \frac{1}{n}\sum_i^n f(x_i)\frac{e^{\lambda x_i}}{\lambda}, \{x_i\} \sim \mathcal{E}(\lambda)$$

Central limit theorem (section 2.10)

Let $X_1, X_2, ...$ be iid random variables with mean $E(X_i) = \mu$ and variance $Var(X_i) = \sigma^2$. For $S_n = \sum_i^n (X_i - \mu)$, what are its mean and variance?

Central limit theorem (section 2.10)

Let $X_1, X_2, \ldots$ be iid random variables with mean $E(X_i) = \mu$ and variance $Var(X_i) = \sigma^2$. For $S_n = \sum_i^n (X_i - \mu)$, what are its mean and variance?

Theorem 2.52 Central Limit Theorem (CLT) If $X_1, X_2, \ldots$ are iid random variables with mean $E(X_i) = \mu$ and variance $Var(X_i) = \sigma^2$, then the distribution of the standardized sum converges to the standard normal distribution:

$$\frac{\sum_i^n (X_i - \mu)}{\sigma\sqrt{n}} \to \mathcal{N}(0,1) \quad \text{as } n \to \infty$$