# Math 40 Probability and Statistical Inference Winter 2021 

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Lecture 7: Transformations and the delta method

## Transformations (2.12)

Goal For a known random variable $X$, we want to find the distribution of $Y=g(X)$, a transformation of $X$.
Example $Y=X^{2}$ where $X \sim \mathcal{N}\left(\mu, \sigma^{2}\right)$, the $\chi^{2}$ distribution (cf.
Exercise 2.7.4, Example 4.18)
Our approach Find the cdf of the transformed random variable. If you want the density, take the derivative of the cdf.

1. $F_{Y}(Y \leq y)=\operatorname{Pr}(Y \leq y)$
2. Represent the right hand side as a probability of $X$,

$$
\operatorname{Pr}(Y \leq y)=\operatorname{Pr}(g(X) \leq y)
$$

3. The density of $Y, f_{Y}(y)$, is given by $\frac{d F_{Y}(y)}{d y}$.

Note If $g$ is invertible, $f_{Y}(y)=\frac{1}{\left|g^{\prime}(y)\right|} f_{X}\left(g^{-1}(y)\right)$.

## Transformations

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pdf: $f_{Y}(y)=\frac{d \sqrt{y}}{d y}=\frac{1}{2 \sqrt{y}}$

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$=\operatorname{Pr}(-\sqrt{y} \leq X \leq \sqrt{y})=\Phi\left(\frac{\sqrt{y}-\mu}{\sigma}\right)-\Phi\left(\frac{-\sqrt{y}-\mu}{\sigma}\right)$ pdf: $f_{Y}(y)=\frac{d \sqrt{y}}{d y}=\frac{1}{2 \sigma \sqrt{y}} \phi\left(\frac{\sqrt{y}-\mu}{\sigma}\right)+\frac{1}{2 \sigma \sqrt{y}} \phi\left(\frac{-\sqrt{y}-\mu}{\sigma}\right)$

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Example 2.66 Find the normalizing transformation of the uniform distribution on $(0,1)$. That is, find the transformation $Z=g(Y)$, $Y \sim \mathcal{R}(0,1)$, such that the distribution of $Z$ is a normal distribution.

## Transformations

Example 2.66 Find the normalizing transformation of the uniform distribution on $(0,1)$. That is, find the transformation $Z=g(Y)$, $Y \sim \mathcal{R}(0,1)$, such that the distribution of $Z$ is a normal distribution.
Solution (without using ODE) In HW2 \#1, we saw that $Y=F(Z)$ is uniform. If $Z$ is the standard normal, $F(Z)=\Phi(Z)$, and thus $Z=\Phi^{-1}$ transforms the uniform distribution $Y$ into the standard normal. Therefore, we have $g=\Phi^{-1}$.

## Transformations

Example 2.67 Find the normalizing transformation of the exponential distribution with the rate parameter $\lambda$. That is, find $g$ such that $Y=g(X)$ becomes a normal distribution where $X \sim \mathcal{E}(\lambda)$.

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Solution (without using ODE; exercise 2.12.8)

1. $Y=F(X)$ is uniform (HW2 \#1).
2. From the previous example, $Z=\Phi^{-1}(Y)$ is the standard normal.
3. So, we have $Z=g(X)=\Phi^{-1}(F(X))$ where $F$ is the cdf of the exponential distribution with $\lambda$. That is,

$$
g(X)=\Phi^{-1}\left(1-e^{-\lambda X}\right)
$$

## Delta methods (2.12.1)

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$$
\sigma_{Y}^{2} \approx\left|g^{\prime}(\mu)\right|^{2} \sigma_{X}^{2}
$$

where $\mu=E[X]$.

## Delta methods (2.12.1)

Example 2.68 The radius of the circle is measured with $1 \%$ error (i.e. the standard deviation is $0.01 \times \mu$ where $\mu$ is the true radius value). What percentage measurement error does it imply for the area?

## Delta methods (2.12.1)

Example 2.68 The radius of the circle is measured with $1 \%$ error (i.e. the standard deviation is $0.01 \times \mu$ where $\mu$ is the true radius value). What percentage measurement error does it imply for the area?
Solution $A=g(R)=\pi R^{2}$. Therefore

$$
\sigma_{A}=\left|g^{\prime}(\mu)\right| \sigma_{R}=2 \pi \mu \times 0.01 \times \mu=0.02 \times \pi \mu^{2}
$$

As $\pi \mu^{2}$ is the true area value, we conclude that the relative measurement error is $2 \%$.

