Math 40 Probability and Statistical Inference

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Lecture 11: Joint density upon transformation (3.6), Optimal portfolio allocation (3.8)

Joint density upon transformation (3.6)

Review of vector calculus

Let's assume that there is a map from $X = (X_1, X_2)$ to $Y = (Y_1, Y_2)$, that is,

$$Y = h(X)$$

or

$$Y_1 = h_1(X_1, X_2), Y_2 = h_2(X_1, X_2)$$

From vector calculus, we know that

$$dy_1 dy_2 = \left| det \left(\frac{\partial h}{\partial X} \right) \right| dx_1 dx_2$$

where $\frac{\partial h}{\partial X}$ is the Jacobian matrix of h

$$\frac{\partial h}{\partial X} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{pmatrix}$$

Joint density upon transformation (3.6)

Review of vector calculus

If h is invertible, say $h^{-1} = g$, then

$$X = g(Y) = (g_1(Y_1, Y_2), g_2(Y_1, Y_2))$$

and

$$dx_1 dx_2 = \left| det \left(\frac{\partial g}{\partial y} \right) \right| dy_1 dy_2$$

Joint density upon transformation (3.6)

Now, we have two bivariate random variables, X and Y, and we know the density of X, i.e., the joint density of X_1 and X_2, say $f_X(x_1, x_2)$.
Then,

$$\int f_X(x_1, x_2) dx_1 dx_2 = \int f_X(g_1(y_1, y_2), g_2(y_1, y_2)) |det\left(\frac{\partial g}{\partial y}\right)| dy_1 dy_2$$

Thus, the probability density of $Y = (Y_1, Y_2)$ is given by

$$f_Y(y_1, y_2) = f_X(g_1(y_1, y_2), g_2(y_1, y_2)) |det\left(\frac{\partial g}{\partial y}\right)|$$

Joint density upon transformation (3.6)

Example 3.62 (warning: your textbook has several typos) Joint density of sum and difference. Random variables X_1 and X_2 are iid with density $f(x)$. Find the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

Joint density upon transformation (3.6)

Example 3.62 (warning: your textbook has several typos) Joint density of sum and difference. Random variables X_1 and X_2 are iid with density $f(x)$. Find the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

- The map from Y to X, that is g in the previous slide, is

$$g(Y) = (\frac{Y_1 + Y_2}{2}, \frac{Y_1 - Y_2}{2})$$

- The Jacobian matrix $\frac{\partial g}{\partial Y}$ is given by

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

- Therefore, $|det\left(\frac{\partial g}{\partial Y}\right)| = \frac{1}{2}$

Joint density upon transformation (3.6)

Example 3.62 (warning: your textbook has several typos) Joint density of sum and difference. Random variables X_1 and X_2 are iid with density $f(x)$. Find the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

- The density of X is $f_X(x_1, x_2) = f(x_1)f(x_2)$.
- $f_X(g_1(y_1, y_2), g_2(y_1, y_2)) = f(\frac{y_1+y_2}{2})f(\frac{y_1-y_2}{2})$
- Finally, we have

$$f_Y(y_1, y_2) = \frac{f(\frac{y_1+y_2}{2})f(\frac{y_1-y_2}{2})}{2}$$

Joint density upon transformation (3.6)

Example 3.63 Random variables X_1 and X_2 have independent exponential distributions with rate parameter λ_1 and λ_2. Find the bivariate density of $Y_1 = X_1$ and $Y_2 = X_2/(aX_1)$ where $a > 0$.

Joint density upon transformation (3.6)

Example 3.63 Random variables X_1 and X_2 have independent exponential distributions with rate parameter λ_1 and λ_2. Find the bivariate density of $Y_1 = X_1$ and $Y_2 = X_2/(aX_1)$ where $a > 0$.

- The map from Y to X, that is g, is

$$g(Y) = (Y_1, aY_1Y_2)$$

- The Jacobian matrix $\frac{\partial g}{\partial Y}$ is given by

$$\begin{pmatrix} 1 & 0 \\ ay_2 & ay_1 \end{pmatrix}$$

- Therefore, $|det\left(\frac{\partial g}{\partial Y}\right)| = ay_1$ ($\because y_1$ is nonnegative).

Joint density upon transformation (3.6)

Example 3.63 Random variables X_1 and X_2 have independent exponential distributions with rate parameter λ_1 and λ_2. Find the bivariate density of $Y_1 = X_1$ and $Y_2 = X_2/(aX_1)$ where $a > 0$.

- The density of X,$f_X(x_1, x_2)$, is $\lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}$.
- $f_X(g_1(y_1, y_2), g_2(y_1, y_2)) = \lambda_1 \lambda_2 e^{-\lambda_1 y_1 - \lambda_2 a y_1 y_2}$
- Finally, we have

$$f_Y(y_1, y_2) = \lambda_1 \lambda_2 a y_1 e^{-\lambda_1 y_1 - \lambda_2 a y_1 y_2}$$

Optimal portfolio allocation (3.8)

- There are two random variables, X and Y, with the same mean μ but with different variances σ_X^2 and σ_Y^2 respectively.
- Q: By taking a combination of X and Y, that is, $\alpha X + (1 - \alpha)Y$, is it possible to have a lower variance than σ_X^2 and σ_Y^2?
- If X and Y represent the returns of two different stocks, this decision process is related to a portfolio design. The expectation is the expected return of each stock, while the variance represents the volatility (or risk or uncertainty).
- The goal of the portfolio design is to minimize the volatility.

Optimal portfolio allocation (3.8)

- From the properties of variance and covariance,

$$Var(\alpha X + (1-\alpha)Y) = \alpha^2\sigma_X^2 + 2\alpha(1-\alpha)\rho\sigma_X\sigma_Y + (1-\alpha)^2\sigma_Y^2$$

where ρ is the correlation between X and Y.

- The optimal α that minimizes the variance is given by

$$\alpha_{opt} = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}$$

- When $\alpha = \alpha_{opt}$, the minimum variance (or risk) is

$$R_{min} = \frac{(1-\rho^2)\sigma_X^2\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}$$

Optimal portfolio allocation (3.8)

If $\rho = 0$, that is, X and Y are uncorrelated,

- $$\alpha_{opt} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$$

  and
- the minimum variance (or risk) is

$$R_{min} = \frac{\sigma_X^2 \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$$

Optimal portfolio allocation (3.8)

In fact, the minimum variance $R_{min} = \frac{(1-\rho^2)\sigma_X^2\sigma_Y^2}{\sigma_X^2+\sigma_Y^2-2\rho\sigma_X\sigma_Y}$, as a function of ρ (this means, everything is known except ρ), is an increasing function.

This implies that *a portfolio using* **negatively** *correlated stocks has a lower risk than other portfolios using uncorrelated or positively correlated stocks.*

Optimal portfolio allocation (3.8)

If you are interested in the case when X and Y have different expected values, and a different interpretation of volatility, read section 3.8.3 and 3.8.4.