Math 42, Winter 2017

Homework set 3, due Wed Jan 25

Collaboration and discussion of the problems is permitted, and even recommended. But you should write up and hand in your own solutions independently.

1. Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ with
$$\gamma(t) = \langle 3t - t^3, 3t^2, 3t + t^3 \rangle$$
Note that γ is *not* a unit speed curve.
   (a) Find the unit tangent vector $\mathbf{T}$ at the point $\gamma(t)$.
   (b) Find the curvature κ and principal normal vector $\mathbf{N}$ at the point $\gamma(t)$.
   Hint. To find $d\mathbf{T}/ds$ use the chain rule
$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$
   (c) Find the binormal $\mathbf{B}$ at point $\gamma(t)$.
   (d) Find the torsion τ at point $\gamma(t)$.

2. If $\gamma(t)$ is conceived as the trajectory of a moving point particle in $\mathbb{R}^3$, then $\dot{\gamma}(t)$ is the velocity vector at time t, and $\ddot{\gamma}(t)$ the acceleration vector. Prove that the tangential and normal components of acceleration are given by
$$\ddot{\gamma} = \frac{dv}{dt}\mathbf{T} + \kappa v^2 \mathbf{N}$$
Here $v = \|\dot{\gamma}\|$ is the speed of the particle, and $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent vector and principal normal vector respectively.

3. The tangent vector of the helix $\gamma(t) = \langle a\cos t, a\sin t, bt\rangle$ is $\dot{\gamma}(t) = \langle -a\sin t, a\cos t, b\rangle$. The dot product of $\dot{\gamma}$ with the unit vector $\mathbf{u} = \langle 0, 0, 1\rangle$ is $\dot{\gamma} \cdot \mathbf{u} = b$. Therefore if θ is the angle between the tangent vector $\dot{\gamma}(t)$ and the unit vector $\mathbf{u}$, then $\cos\theta = b$. It follows that the angle θ is constant.

   Definition 1 *A* generalized helix *is a curve in $\mathbb{R}^3$ with curvature $\kappa \neq 0$, all of whose tangent vectors make a constant angle with a fixed unit vector* $\mathbf{u}$.

   Suppose $\gamma(s)$ is a unit speed generalized helix with Frenet frame $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}(s)$. Let θ be the constant angle between $\mathbf{T}$ and $\mathbf{u}$, i.e. $\mathbf{T} \cdot \mathbf{u} = \cos\theta$.
   (a) Prove that $\mathbf{N} \perp \mathbf{u}$.
   (b) Prove that $\mathbf{B} \cdot \mathbf{u} = \pm \sin\theta$.
   (c) Prove that for a generalized helix the quotient τ/κ is the same at every point $\gamma(s)$.
   Hint. Take the derivative of $\mathbf{N} \cdot \mathbf{u} = 0$ and use the formula for $\dot{\mathbf{N}}$.
   (d) Does there exist a generalized helix with $\tau/\kappa = 3$ that is not congruent to a (circular, ordinary) helix? Explain.