Math 42, Winter 2017

Homework set 5, due Wed Feb 8

Collaboration and discussion of the problems is permitted, and even recommended. But you should write up and hand in your own solutions independently.

1. Consider the quadratic surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = ax^2+2bxy+cy^2\}$, where a, b, c are three constants. The single chart $\sigma : \mathbb{R}^2 \to S$ with $\sigma(u, v) = (u, v, au^2 + 2buv + cv^2)$ parametrizes all of S.

   Calculate the First Fundamental Form
   $$\|\mathbf{v}\|^2 = E\lambda^2 + 2F\lambda\mu + G\mu^2 \qquad \mathbf{v} = \lambda\sigma_u + \mu\sigma_v$$
   for the chart σ. Note that E, F, G are functions of the parameters u, v.

2. This problem is about the unit sphere $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let σ be the chart $\sigma : \mathbb{R}^2 \to S$ that was given on the previous homework set, [1]
   $$\sigma(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)$$
   Let f be the smooth map $f : S \to S$ given in x, y, z coordinates as
   $$f(x, y, z) = (x^2 - y^2, 2xy, z\sqrt{2 - z^2})$$
   (a) Show that if $(x, y, z) \in S$ then also $f(x, y, z) \in S$, i.e. f is indeed a map $S \to S$.

   (b) Show that if P is a point on the equator of S, then the derivative $D_Pf : T_PS \to T_PS$ is an invertible map.

   (c) If Q is the South Pole $Q = (0, 0, -1)$, what is the derivative D_Qf of f at Q?

   (d) Calculate the First Fundamental Form of the sphere in the surface patch σ, i.e. find formulas for the coefficients E, F, G in the quadratic form $E\lambda^2 + 2F\lambda\mu + G\mu^2$ (in terms of u, v).

   (e) Use the First Fundamental Form to find the length of the closed curve γ on the sphere S that corresponds to the circle $u^2 + v^2 = r^2$ of radius $r > 0$ in the chart domain.

[1] On Homework set 4, σ was referred to as σ_1.