CONTOUR INTEGRATION

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In real calculus, we learn about differentiation and integration. In multivariable (real) calculus, we learn about several different types of integration: for example, there is multiple integration over subsets of $\mathbb{R}^n$, and then there is line and surface integration, which takes place over curves in $\mathbb{R}^n$ and surfaces in $\mathbb{R}^3$. In complex analysis, we will primarily be interested in a complex version of line integration, which is frequently called *contour integration*.

1. Parameterization of curves

Recall from vector calculus the notion of a line integral: given a function $f(x, y)$ in $\mathbb{R}^2$, say, and a curve C in $\mathbb{R}^2$ parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \leq t \leq b$, the line integral of f along C is equal to the definite integral

$$\int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\,dt.$$

There is a related expression which gives the value of a line integral of a vector field along a curve. In complex analysis we will be interested in computing yet another type of line/contour integral, whose definition uses the fact that there is a natural multiplicative structure on $\mathbb{C}$.

Let $z : [a, b] \to \mathbb{C}$ be a function from a closed interval of $\mathbb{R}$ to $\mathbb{C}$. We can plot the graph of z by sketching its image in $\mathbb{C}$; in general this will look like a curve. If we write $z(t) = x(t) + y(t)i$, the derivative of $z(t)$ is just $z'(t) = x'(t) + y'(t)i$, assuming both x, y are differentiable as real functions. We define a *parameterized curve* to be any function $z : [a, b] \to \mathbb{C}$. A parameterized curve is said to be *smooth* if $z : [a, b] \to \mathbb{C}$ is a function such that $z(t)$ is differentiable, $z'(t)$ is continuous on $[a, b]$, and $z'(t) \neq 0$ at any t. (At the endpoints $t = a, b$, this derivative is to be interpreted as a one-sided derivative.)

More generally, a piecewise (parameterized) smooth curve is any function $z : [a, b] \to \mathbb{C}$ for which there exist a finite number of real numbers $a = t_0 < t_1 < \ldots < t_n = b$ such that z restricted to each closed interval $[t_i, t_{i+1}]$ is a parameterized smooth curve.

A parameterized curve $z : [a, b] \to \mathbb{C}$ is said to be *closed* if $z(a) = z(b)$. A parameterized curve is said to be closed if z is not self-intersecting except possibly at its endpoints, ie, if $z(t_1) = z(t_2)$, then $t_1 = t_2$ or $t_1, t_2 = a, b$.

Examples.

- An important class of curves which we will encounter repeatedly over the rest of the class are circles in $\mathbb{C}$. Let C be a circle of radius r centered at the point z_0. Then C can be parameterized by $z(t) = z_0 + re^{it}, 0 \le t \le 2\pi$. Since $z(t) = z_0 + r(\cos t + i \sin t)$, its derivative is given by $z'(t) = -r \sin t + i(r \cos t) = rie^{it}$. We can see that this parameterization is smooth, since $z'(t)$ is continuous and never equals 0 anywhere. Furthermore, this parameterization is simple and closed. This parameterization will occur so frequently in this class that you should memorize it!
- Another important (though not quite as important) class of curves which we will encounter are line segments connecting two (distinct) complex numbers z_0 to z_1. If we want to parameterize the line segment joining z_0, z_1 with a function $z(t)$ on the interval $[0, 1]$ where $z(0) = z_0, z(1) = z_1$, then we can let $z(t) = z_0 + t(z_1 - z_0)$. One quickly checks $z'(t) = z_1 - z_0 \neq 0$ is continuous, so this parameterization is smooth. This parameterization is simple, but not closed.
- Piecewise smooth curves appear frequently in complex analysis. In many cases they consist of line segments and circular arcs joined in certain arrangements. For example, polygons like triangles, rectangles, etc. are all piecewise smooth curves. Similarly, semicircles, or wedges of circles, are also piecewise smooth curves.
- Notice that the graph of a parameterized curve does not uniquely determine the parameterization. For example, if $C = S^1$ is the unit circle centered at the origin, then two different parameterizations whose graph is C are $z_1(t) = e^{it}, 0 \le t \le 2\pi$, and $z_2(t) = e^{2it}, 0 \le t \le \pi$.
- When writing parameterizations of curves, you should get in the habit of not just indicating the function $z(t)$ but also the domain $[a, b]$ of z. Many sources will omit the domain if it is obvious from the context what the domain is, but do not do this until you are absolutely sure of what you are doing!
- If $z(t)$ is a simple closed curve in $\mathbb{C}$, then a seemingly intuitively obvious but surprisingly difficult theorem to prove (the Jordan curve theorem) states that the graph of this curve divides $\mathbb{C}$ into two connected components: an interior and an exterior. We say that $z(t)$ has *positive orientation* if the interior of this curve is on its left as t increases; an equivalent formulation is that $z(t)$ induces a counterclockwise orientation on its graph. For example, $z(t) = e^{it}, 0 \le t \le 2\pi$ is a parameterization of S^1 which has positive orientation, while e^{-it} has negative orientation.

Even though the graph of a parameterized curve does not uniquely determine the underlying parameterization, we can define a notion of when two parameterized curves are equivalent. Suppose $z_1 : [a, b] \to \mathbb{C}, z_2 : [c, d] \to \mathbb{C}$ are two parameterized smooth curves. Then they are *equivalent* if there exists a function $t(s) : [c, d] \to [a, b]$ such that t' is continuous, $t(c) = a, t(d) = b$, $t'(s) > 0$ everywhere, and $z_1(t) = z_2(t(s))$. One can check that this definition of equivalence is actually an equivalence relation (ie, is reflexive, symmetric, and transitive).

The equivalence class of a given parameterized curve consists of parameterized curves whose graphs are all the same curve in $\mathbb{C}$ and share the same *orientation* on that curve: that is, they all share the same starting point and endpoint. Indeed, the

$t(c) = a, t(d) = b$ conditions ensure that the starting and end points are the same, and the $t'(s) > 0$ condition ensures that two equivalent curves traverse the graph in the same direction and trace out each point the same number of times.

Given a parameterized curve $z(t) : [a, b] \to \mathbb{C}$, there is a canonical way to obtain a parameterized curve whose graph is the same as $z(t)$ but has opposite orientation: namely, consider the function $z^- : [a, b] \to \mathbb{C}$ defined by $z^-(t) = z(b + a - t)$. z^- and z have the same graph, but opposite starting and end points.

Examples.

- The two curves $z_1(t) = e^{it}, 0 \le t \le 2\pi$, and $z_2(t) = e^{2it}, 0 \le t \le \pi$ are equivalent. Indeed, their graphs are both S^1, and a re-parameterization of z_1 to z_2 is given by $t(s) = s/2$; ie, $z_2(t) = e^{2it} = z_1(2t)$.
- The two curves $z_1(t) = e^{it}, 0 \le t \le 2\pi$ and $z_2(t) = e^{-it}, 0 \le t \le 2\pi$ trace out the same curve, but in opposite orientations.
- The two curves $z_1(t) = e^{it}, 0 \le t \le 2\pi$, and $z_2(t) = e^{2it}, 0 \le t \le 2\pi$, are not equivalent. The intuitive way to see this is the fact that z_1 traces out S^1 once, while z_2 does so twice. It is not too difficult to make this more precise.

2. Contour integration

With an understanding of how to parameterize curves, we can now define contour integration. Let γ be a smooth curve parameterized by $z(t), a \le t \le b$, and let $f(z)$ be a continuous (or possibly piecewise-continuous) complex function defined on γ. Then the *integral* of f along γ is defined to be

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt,$$

if the definite integral on the right hand side exists. (The integrand is a complex function of a real variable, and we integrate it by integrating the real and imaginary parts separately.) We will soon show that this definition is independent of the parameterization for γ, as long as we restrict to equivalent parameterizations (ie, parameterizations giving the same orientation), but before doing so we will calculate a few simple examples.

Examples.

- Suppose $f(z) = z$, and we want to integrate along the circle $z(t) = e^{it}, 0 \le t \le 2\pi$. Then $z'(t) = ie^{it}$, so

$$\int_\gamma z\, dz = \int_0^{2\pi} z(t)\cdot z'(t)\, dt = \int_0^{2\pi} ie^{2it}\, dt = \int_0^{2\pi} i(\cos 2t + i \sin 2t)\, dt = \int_0^{2\pi} -\sin 2t + i\cos 2t\, dt.$$

  We need to compute the two integrals $\int_0^{2\pi} -\sin 2t\, dt, \int_0^{2\pi} \cos 2t\, dt$; these are both equal to 0, so the original integral is equal to $0 + 0i = 0$.
- Suppose $f(z) = \overline{z}$, and γ is the line segment connecting 0 to $1 + i$. Then we can parameterize γ by using $z(t) = t + ti, 0 \le t \le 1$, so $z'(t) = 1 + i$. Then

$$\int_\gamma \overline{z}\,dz = \int_0^1 \overline{z(t)}\cdot z'(t)\,dt = \int_0^1 (t-ti)(1+i)\,dt = \int_0^1 (t+t)+i(t-t)\,dt = \int_0^1 2t\,dt = 1.$$

- The next calculation is probably the most important contour integral we will do in this class. Let $f(z) = 1/z$, and let $z(t) = e^{it}, 0 \le t \le 2\pi$ parameterize the unit circle with positive orientation. Then $z'(t) = ie^{it}$, and

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

In actuality we will not compute too many explicit line integrals; rather, it is their properties, especially when integrating holomorphic functions, which are of primary interest. The first property we prove is that the line integral is independent of parameterization:

Proposition 1. *If $z_1(t), z_2(t)$ are equivalent parameterizations of a smooth curve γ, they both yield the same value for the integral of any function $f(z)$ along γ.*

Proof. Suppose $t(s) : [c,d] \to [a,b]$ reparameterizes $z_2(t), c \le t \le d$, to $z_1(t), a \le t \le b$, so that $z_1(t(s)) = z_2(s)$. Then using the change of variables $t = t(s)$, we get

$$\int_a^b f(z_1(t))z_1'(t)\,dt = \int_c^d f(z_1(t(s))z_1'(t(s))t'(s)\,ds.$$

However, since $z_1(t(s)) = z_2(s)$, we have $z_1'(t(s))t'(s) = z_2'(s)$ by the chain rule, so the integral above is just equal to

$$\int_c^d f(z_2(s)) \cdot z_2'(s)\,ds.$$

□

This property probably sounds familiar, because there is an identical property that one can prove using almost exactly the same idea when showing that line integrals of scalar functions or vector fields are independent of choice of parameterization of the underlying curve. As an exercise, for each of the examples above, you can try reparameterizing the curve with a different parameterization, calculate the line integral using that parameterization, and see that you get the same result.

More generally, suppose γ is a piecewise smooth curve. Then we can define the line integral of $f(z)$ over γ by breaking apart γ into its smooth components $\gamma_1, \gamma_2, \ldots, \gamma_n$, and then integrating $f(z)$ over each smooth curve γ_i and adding the results together. Again, this is exactly identical to how line integrals in vector calculus are calculated over piecewise smooth curves.

Example. Suppose γ is the polygonal path from 0 to 1 and then 1 to $1+i$. Let γ_1, γ_2 be these two paths. Let $f(z) = z^2$. Then

$$\int_\gamma z^2\,dz = \int_{\gamma_1} z^2\,dz + \int_{\gamma_2} z^2\,dz.$$

We can parameterize γ_1 using $z_1(t) = t, 0 \le t \le 1$, so $z'(t) = 1$, and

$$\int_{\gamma_1} z^2\,dz = \int_0^1 t^2\,dt = \frac{1}{3}.$$

Similarly, we can parameterize γ_2 using $z_2(t) = 1 + it, 0 \le t \le 1$, so $z'(t) = i$, and

$$\int_{\gamma_2} z^2\,dz = \int_0^1 (1+it)^2 i\,dt = \int_0^1 -2t + i(1-t^2)\,dt = -1 + \frac{2}{3}i.$$

Therefore $\int_\gamma z^2\,dz = -2/3 + 2i/3$.

There are many properties of integrals from single variable calculus or vector calculus which carry over to contour integrals of complex functions, and we omit the simpler proofs because of their similarity to the proofs of the corresponding real statements:

- Suppose c is any complex number. Then $\int_\gamma cf\,dz = c\int_\gamma f\,dz$.
- Suppose f, g are both continuous. Then $\int_\gamma (f+g)\,dz = \int_\gamma f\,dz + \int_\gamma g\,dz$. Taken together, this property with the previous property show that contour integration is linear over $\mathbb{C}$.
- Suppose γ^- is γ with opposite orientation. Then $\int_\gamma f\,dz = -\int_{-\gamma} f\,dz$; in other words, reversing the orientation of the curve γ flips the sign of the integral. For example, recall that if γ is S^1 with positive orientation, then $\int_\gamma 1/z\,dz = 2\pi i$. We can parameterize γ^- using $z(t) = e^{-it}, 0 \le t \le 2\pi$, so

$$\int_{\gamma^-} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{-it}} \cdot -ie^{-it}\,dt = -2\pi i.$$

- Suppose γ is parameterized by $z(t), a \le t \le b$. Then the integral $\int_a^b |z'(t)|\,dt$ is just the arc length of γ, as the real calculus formula for arc length shows.
- One property of definite integrals in one variable which does not carry over (at least not in a naive way) is u-substitution. More specifically, do not make substitutions of the form $u = u(z)$ in integrals. There is a way to try to rigorously justify an analogue of u-substitution for contour integrals, but it requires more knowledge than we have now.

All of the above properties will be repeatedly used for the rest of the class. The following property will also be used, but because it is not as obvious how to prove (although intuitively the conclusion makes sense), we will give a rigorous proof:

Proposition 2. *Suppose γ is a piecewise smooth curve and has arc length L. Let $f(z)$ be a continuous function on a set containing γ, and suppose* $\sup_{z\in\gamma} |f(z)| = M$. *(More generally, we can allow M to be any upper bound for $|f(z)|$ over $z \in \gamma$, but the smaller M is the better the conclusion of this theorem will be.) Then*

$$\left|\int_\gamma f(z)\,dz\right| \le ML.$$

Proof. Using the definition for contour integrals,

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right|,$$

where $z(t), a \leq t \leq b$, is a parameterization for γ. A property from real calculus says that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

An analogue of this property holds where we permit the integrand to be a complex-valued function of a real variable (this will be on the next homework set); namely,

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt,$$

where $f(t) = x(t) + y(t)i$ is a complex function of a real variable. Applying this to the integral we are interested in,

$$\left| \int_a^b f(z(t))z'(t)\, dt \right| \leq \int_a^b |f(z(t))z'(t)|\, dt = \int_a^b |f(z(t))||z'(t)|\, dt \leq M \int_a^b |z'(t)|\, dt = ML,$$

where we use the fact that $|f(z(t))| \leq M$ for all $a \leq t \leq b$ by the definition of M. □

This property will be very useful in situations where we are unable to explicitly evaluate a particular contour integral, but will need estimates on its size. Sometimes this proposition provides strong enough estimates and sometimes it does not.

The next theorem is a contour integral version of the familiar fundamental theorem of calculus for definite integrals of a single variable. It is also reminiscent of the fundamental theorem of calculus for line integrals (which really is a generalization of the ordinary fundamental theorem of calculus), and the method of proof of the complex analysis version is essentially identical to the line integral version. However, we will shortly see that you need to be careful with its use!

Suppose f is a continuous complex function defined on an open set Ω of $\mathbb{C}$. We call another complex function F a *primitive* for f on Ω if F is holomorphic on Ω and $F'(z) = f(z)$ in Ω. In other words, a primitive is the complex analysis version of an antiderivative.

Theorem 1. *Suppose F is a primitive for f on Ω and γ is a smooth curve completely contained in Ω whose starting point is w_1 and end point is w_2. Then*

$$\int_\gamma f(z)\, dz = F(w_2) - F(w_1).$$

Proof. Let $z(t), a \leq t \leq b$, be a smooth parameterization for γ. We have

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))z'(t)\, dt = \int_a^b F'(z(t))z'(t)\, dt.$$

One can prove that a version of the chain rule applies to the composition $F \circ z$; namely, if F is holomorphic and $z(t)$ is C^1, then

$$\frac{d}{dt} F(z(t)) = F'(z(t))z'(t).$$

(This will be on next week's homework.) Taking this fact for granted, we have

$$\int_a^b F'(z(t))z'(t)\,dt = \int_a^b \frac{d}{dt}F(z(t))\,dt = F(z(b)) - F(z(a)),$$

where in the last equality we just use the fact that $\int_a^b f'(t)\,dt = f(b) - f(a)$ for real functions, applied to the real and imaginary parts of $F(z(t))$. □

Corollary 1. *The theorem above holds if γ is piecewise-smooth instead of smooth.*

Proof. Just break γ up into its smooth pieces $\gamma_1, \gamma_2, \ldots, \gamma_n$. Let γ_i have starting point w_i and endpoint w_{i+1}. Then

$$\int_\gamma f(z)\,dz = \sum \int_{\gamma_i} f(z)\,dz = \sum F(w_{i+1}) - F(w_i).$$

This is just a telescoping sum equal to $F(w_{n+1}) - F(w_1)$, as desired. □

The following two corollaries are complex-analysis versions of path-independence of line integrals of conservative vector fields.

Corollary 2. *Suppose f has a primitive on an open set Ω containing the piecewise smooth closed curve γ. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Corollary 3. *Suppose γ_1, γ_2 are two paths completely contained in an open set Ω with the same starting and end point. If $f(z)$ has a primitive in Ω then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

This theorem and its corollaries are surprisingly useful; they makes evaluating a lot of integrals much easier. In particular, if the integrand of a contour integral has a primitive (for example, $f(z) = e^z$ or a polynomial), this theorem can save a lot of work. On the other hand, you must be ABSOLUTELY SURE that $f(z)$ has a primitive on all of γ, as the last example below illustrates:

Examples.

- Since e^z is its own primitive on all of $\mathbb{C}$, the integral of e^z across any closed curve in $\mathbb{C}$ is 0. In a similar vein, suppose $z(t) = t^2 + e^{\sin t}, 0 \le t \le \pi/2$, parameterizes a curve γ. Integrating e^z along γ directly would be fairly painful, but if we note that its starting point is $z(0) = 0 + i$ and end point is $z(\pi/2) = (\pi/2)^2 + ei$, then this integral equals $e^{(\pi/2)^2+ei} - e^i$.
- The integral of any polynomial along any closed curve is 0, since a polynomial has a primitive on all of $\mathbb{C}$. (For example, z^n has the primitive $z^{n+1}/(n+1)$.)
- Notice that $1/z$ is continuous when $z \neq 0$. Nevertheless, we claim that $1/z$ has no primitive on all of $\mathbb{C} - 0$. In particular, there is no way to extend $\log x$ defined on positive reals to the punctured complex plane. Indeed, if $1/z$ did have a primitive on $\mathbb{C} - 0$, then $\int_{S^1} 1/z\,dz$ would equal 0, but we already saw that this integral equals $2\pi i$ by direct calculation. So be ABSOLUTELY

SURE that f has a primitive on all of γ if you want to apply the above theorems or any of its corollaries!

We have proven the following corollaries essentially using other means (in particular, one can apply the solution to the problem on functions with constant real or imaginary part on a previous homework set), but we provide a different proof using these new ideas:

Corollary 4. *Suppose $f(z)$ is holomorphic on a region (ie, open connected set) Ω, and suppose $f'(z) = 0$ for all $z \in \Omega$. Then $f(z)$ is constant on Ω.*

Proof. Pick any z_0 in Ω. Since Ω is connected, we can find a path γ connecting z_0 to z. As a matter of fact, we can select γ to be piecewise-smooth; for example, recall that z_0 and z can actually be connected via a polygonal path, which is obviously piecewise-smooth. Then

$$\int_\gamma f'(z)\,dz = f(z) - f(z_0) = 0,$$

since f is a primitive for f' in Ω, and $f'(z) = 0$ everywhere. Therefore $f(z) = f(z_0)$; since z was arbitrary this means that f is constant and equal to $f(z_0)$ on all of Ω. $\square$