The Complex Exponential Function

Theorem

1. *We have* $e^z = 1$ *if and only if* $z = 2\pi i k$ *for some* $k \in \mathbb{Z}$.
2. *We have* $e^z = e^w$ *if and only if* $z = w + 2\pi i k$ *for some* $k \in \mathbb{Z}$.

Corollary

The complex exponential function $f(z) = e^z$ *is periodic of period* $2\pi i$. *Hence the behavior of* $z \mapsto e^z$ *is identical in every strip of the form*

$$S_n = \{ z \in \mathbb{C} : (2n-1)\pi < \operatorname{Im}(z) \leq (2n+1)\pi \} \quad \text{for } n \in \mathbb{Z}.$$

More Toys

Definition

For all $z \in \mathbb{C}$ we define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Theorem

$f(z) = \cos(z)$ and $g(z) = \sin(z)$ are entire and periodic of period 2π. Furthermore,

$$\frac{d}{dz}(\cos(z)) = -\sin(z) \quad \textit{and} \quad \frac{d}{dz}(\sin(z)) = \cos(z)$$

for all $z \in \mathbb{C}$.

The Complex Logarithm

Definition

If $z \neq 0$, then we define

$$\log(z) = \ln\big(|z|\big) + i\arg(z) = \{\, \ln\big(|z|\big) + iy : y \in \arg(z) \,\}.$$

Definition

We define the principal branch of $\log(z)$ to be the function

$$\operatorname{Log}(z) = \ln\big(|z|\big) + i\operatorname{Arg}(z) \quad \text{for all } z \neq 0.$$