Fields

Definition

A field is a set $\mathbf{F}$ containing at least two elements 0 and 1 equipped with operations $+$ and $\cdot$ such that for all $x, y, z \in \mathbf{F}$ we have $x + y \in \mathbf{F}$ and $x \cdot y = xy \in \mathbf{F}$ and

1)	$x + y = y + x$	1)′	$xy = yx$
2)	$x + (y + z) = (x + y) + z$	2)′	$x(yz) = (xy)z$
3)	$x + 0 = x$	3)′	$x \cdot 1 = x$
4)	there exists $-x$ such that $-x + x = 0$	4)′	if $x \neq 0$ there exists x^{-1} such that $xx^{-1} = 1$, and

5) $x(y + z) = xy + yz$.

Example

Of course our favorite example of a field is the field of rational numbers $\mathbf{Q} = \{ \frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbf{N} \}$. But there are lots of others.

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A Field with Four Elements

Example

Let $\mathbf{F} = \{0, 1, a, b\}$. Then define addition and multiplication as follows

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Then it is possible to show that $\mathbf{F}$ is a field. However in all honesty, it would be tedious beyond belief to check this directly. Fortunately, there are other techniques—from abstract algebra—that allow us to see this from general principles.

Ordered Fields

Definition

We say that a field $\mathbf{F}$ is ordered if there is a subset $P \subset \mathbf{F} \setminus \{0\}$ such that

1. $\mathbf{F}$ is the disjoint union of P, $\{0\}$, and $-P$.
2. If $a, b \in P$, then $a + b \in P$ and $ab \in P$.

We say that $x > 0$ if $x \in P$ and $x < y$ if $y - x \in P$. We call the pair $(\mathbf{F}, P)$, or sometimes $(F, <)$ an ordered field.

Remark

If a is an element in an ordered field, either a is positive, $-a$ is positive, or $a = 0$. Alternatively, given a, b in an ordered field, either $a < b$, $b < a$, or $a = b$.

Example

Let $P = \{ \frac{a}{b} \in \mathbf{Q} : a, b \in \mathbf{N} \}$. Then $(\mathbf{Q}, P)$ is an ordered field that we've held dear to our hearts since grade school.

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