Trigonometric Integrals

- Observe that if we parameterize the positively oriented circle $|z| = 1$ by $z(t) = e^{i\theta}$ for $\theta \in [0, 1]$ then

$$\int_{|z|=1} F(z)\, dz = \int_0^{2\pi} F(e^{i\theta}) i e^{i\theta}\, d\theta.$$

- Furthermore, if $z = e^{i\theta}$ lies on the circle $|z| = 1$, then

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{while} \quad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Formulas

Theorem

Suppose that $U(x,y)$ is a rational function such that $U(\cos(\theta),\sin(\theta))$ is defined for all θ. Then

$$\int_0^{2\pi} U\big(\cos(\theta),\sin(\theta)\big)\,d\theta = \oint_{|z|=1} F(z)\,dz = 2\pi i \sum_{|z|<1} \operatorname{Res}(F;z)$$

where

$$F(z) = U\Big(\frac{1}{2}\Big(z+\frac{1}{z}\Big),\frac{1}{2i}\Big(z-\frac{1}{z}\Big)\Big)\cdot\frac{1}{iz}.$$

Basic Limit Lemma

Theorem (Basic Limit Lemma)

Let C_R^+ be the top half of the positively oriented circle $|z| = R$ from R to $-R$. Suppose that $p(z)$ and $q(z)$ are polynomials such that

$$\deg p(z) + 2 \leq \deg q(z).$$

If $a \geq 0$ and

$$F(z) = \frac{p(z)}{q(z)} e^{iaz},$$

then

$$\lim_{R\to\infty} \int_{C_R^+} F(z)\, dz = 0.$$

Improper Riemann Integrals

Recall that a continuous function on **R** is integrable if $\int_{-\infty}^{\infty} f(x)\,dx$ exists (or some would say converges). This means both the limits

$$L = \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx \quad \text{and} \quad M = \lim_{R\to\infty} \int_{0}^{R} f(x)\,dx$$

exist (and are finite). Then $\int_{-\infty}^{\infty} f(x)\,dx = L + M$. This is not quite the same as saying that the Cauchy Principal Value

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

exists. (Consider $f(x) = x$.) Nevertheless, if f is integrable, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx.$$

Existence of Improper Integrals

Theorem (Comparison Test)

Let f and g be continuous functions on the real line. Suppose that g is integrable and non-negative on $\mathbf{R}$. Then if $|f(x)| \leq g(x)$ for all $x \in \mathbf{R}$, then f is integrable.

Corollary

Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients. Suppose also the $\deg p(x) + 2 \leq \deg q(x)$ and $q(x)$ has no real roots. Then

$$f(x) = \frac{p(x)}{q(x)}$$

is integrable.

Remark

This means that we can drop the "p.v." from our formulas in the next result.

Another Formula

Theorem

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 2 \leq \deg q(z)$ and such that $q(x)$ has no real roots. Suppose that $a \geq 0$ and

$$F(z) = \frac{p(z)}{q(z)} e^{iaz}.$$

Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Bigl(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)\Bigr)$$

and

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Bigl(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)\Bigr). \tag{1}$$

Remark

Note that if $a = 0$ in the above result, then (1) is zero and $2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)$ must be real.

An Improved Formula

Now using Jordan's Lemma in place of the Basic Limit Lemma:

Theorem

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 1 \le \deg q(z)$ and such that $q(x)$ has no real roots. Suppose that $a > 0$ and

$$F(z) = \frac{p(z)}{q(z)} e^{iaz}.$$

Then

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\left(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)\right)$$

and

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\left(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)\right).$$

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