Jordan's Lemma

Theorem (Jordan's Lemma)

Suppose that $p(z)$ and $q(z)$ are polynomials with

$$\deg p(z) + 1 \leq \deg q(z).$$

Let $a > 0$ and

$$F(z) = \frac{p(z)}{q(z)} e^{iaz}.$$

This if C_R^+ is the top half of the circle $|z| = R$ from R to $-R$, then

$$\lim_{R\to\infty} \int_{C_R^+} F(z)\, dz = 0.$$

An Improved Formula

Now using Jordan's Lemma in place of the Basic Limit Lemma:

Theorem

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 1 \leq \deg q(z)$ and such that $q(x)$ has no real roots. Suppose that $a > 0$ and

$$F(z) = \frac{p(z)}{q(z)} e^{iaz}.$$

Then

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Bigl(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)\Bigr)$$

and

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Bigl(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}(F; z)\Bigr).$$

It turns out we can drop the "p.v."s. Trust me.

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Indented Contours

Theorem

Suppose that f has a simple pole at z_0. Let $C_r(\theta_1, \theta_2)$ be the arc of the positively oriented circle $|z - z_0| = r$ parameterized by $z(t) = z_0 + re^{it}$ with $t \in [\theta_1, \theta_2]$. Then

$$\lim_{r \searrow 0} \int_{C_r(\theta_1,\theta_2)} f(z)\, dz = (\theta_2 - \theta_1) i \operatorname{Res}(f; z_0).$$