Last Time

Definition

A series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}$$

is called a Laurent Series about z_0.

Laurent Series 101

Theorem

A Laurent series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \tag{1}$$

either does not converge for any $z \in \mathbf{C}$, *or there are* $0 \leq r \leq R \leq \infty$ *such that* (1) *converges absolutely for all* z *in*

$$A = \{ z \in \mathbf{C} : r < |z - z_0| < R \}.$$

Moreover, the convergence is uniform in any sub-annulus

$$A' = \{ z \in \mathbf{C} : r' \leq |z - z_0| \leq R' \}$$

provided $0 \leq r < r' < R' < R \leq \infty$.

Cauchy Again

Theorem (Cauchy's Integral Formula for an Annulus)

Suppose that f is analytic in

$$A = \{\, z \in \mathbf{C} : 0 \le r < |z - z_0| < R \le \infty \,\}$$

with $r < R$. Let C_ρ be the positively oriented circle $|z - z_0| = \rho$. Suppose $r < \rho_1 < \rho_2 < R$ and $\rho_1 < |z - z_0| < \rho_2$. Then

$$f(z) = \frac{1}{2\pi i} \int_{C_{\rho_2}} \frac{f(\omega)}{\omega - z}\, d\omega - \frac{1}{2\pi i} \int_{C_{\rho_1}} \frac{f(\omega)}{\omega - z}\, d\omega.$$

For Today

Theorem (Laurent's Theorem)

Suppose that f is analytic in

$$A = \{ z \in \mathbf{C} : 0 \leq r < |z - z_0| < R \leq \infty \}$$

with $r < R$. Then there are complex constants a_n and b_j such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$$

for all $z \in A$. Moreover if C is any positively oriented simple closed contour in A with z_0 in its interior, then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - z_0)^{n+1}} \, d\omega \quad \text{and}$$

$$b_j = \frac{1}{2\pi i} \int_C f(\omega)(\omega - z_0)^{j-1} \, d\omega.$$