Harmonic Conjugates

Theorem

Suppose that u is Harmonic in a Domain D.

1. *If v and w are Harmonic conjugates of u in D, then v and w differ by a real constant.*
2. *There may not be a Harmonic conjugate for u in all of D.*
3. *If D is an open disk, then u has a harmonic conjugate in D.*

Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Suppose $p(z)$ is a polynomial with complex coeffieients and $\deg p(z) = n \geq 1$. Then $p(z)$ has at least one complex root. Hence there are complex numbers $a, z_1, \ldots, z_n$ such that

$$p(z) = a(z - z_1) \cdots (z - z_n).$$

Theorem

Suppose that $p(z) \in \mathbb{C}[z]$ has real coefficients and that $\deg p(z) \geq 1$. Then there are real numbers $r_1, \ldots, r_s$ and irreducible quadradics $q_1(z), \ldots, q_k(z)$ such that $\deg p(z) = r + 2k$ and

$$p(z) = a(z - r_1) \cdots (z - r_s) q_1(z) \cdots q_k(z).$$

Partial Fraction Decompositions

Theorem (§3.1, Theorem 2)

Suppose that

$$R(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{a(z - w_1)^{d_1}(z - w_2)^{d_2} \cdots (z - w_s)^{d_s}}$$

is a rational function with $\{ w_1, \ldots, w_s \}$ *distinct and* $\deg p(z) < \deg q(z) = d_1 + \cdots d_s$. *Then*

$$R(z) = r_1(z) + \cdots + r_s(z)$$

with

$$r_k(z) = \frac{A_0^{(k)}}{(z - w_k)^{d_k}} + \frac{A_1^{(k)}}{(z - w_k)^{d_k - 1}} + \cdots + \frac{A_{d_k - 1}^{(k)}}{z - w_k}$$

for complex constants $A_j^{(k)}$.

Example

The theorem on the previous slide says that there are constants A, B, C, and D such that

$$f(z) = \frac{4z+4}{z(z-1)(z-2)^2} = \underbrace{\frac{A}{z}}_{r_1(z)} + \underbrace{\frac{B}{z-1}}_{r_2(z)} + \underbrace{\frac{C}{(z-2)^2} + \frac{D}{z-2}}_{r_3(z)}.$$

It follows that

$$zf(z) = A + z\big(r_2(z) + r_3(z)\big).$$

Hence

$$A = \lim_{z \to 0} zf(z).$$

Example

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$$\frac{4z+4}{z(z-1)(z-2)^2} = -\frac{1}{z} + \frac{8}{z-1} + \frac{6}{(z-2)^2} - \frac{7}{z-2}.$$

Theorem (§3.1, Equation (21))

In general, if $R(z)$ has the form in the Decomposition Theorem, then

$$A_j^{(k)} = \lim_{z\to w_k} \frac{1}{j!}\frac{d^j}{dz^j}\left((z-w_k)^{d_j} R(z)\right).$$