Term-by-Term Differentiation

Theorem

Suppose that f is analytic in $B_R(z_0)$ with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then the Taylor series for the derivative f' in $B_R(z_0)$ is given by term-by-term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{1}$$

Products

Theorem

Suppose that f and g are analytic at z_0 with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n .$$

Then the Taylor series for $h = fg$ about z_0 is given by the Cauchy product

$$h(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} .$$

$\tan(z)$

Example (MacLaurin Series for $\tan z$)

Here is

$$\tan(z) = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \frac{62z^9}{2835} + \frac{1382z^{11}}{155925} + \frac{21844z^{13}}{6081075} + \frac{929569z^{15}}{638512875} + \dots$$

Power Series 101

Theorem

Let

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be a power series about z_0. *Then there is a* R *such that* $0 \leq R \leq \infty$ *and such that*

1. *The series converges absolutely if* $|z - z_0| < R$.
2. *The series converges uniformly on any subdisk*

   $$D_r = \{\, z \in \mathbf{C} : |z - z_0| \leq r \,\}$$

   provided that $0 < r < R$.
3. *The series diverges if* $|z - z_0| > R$.

Uniformly Good

Theorem

Suppose that (f_n) is a sequence of continuous complex-valued functions converging uniformly to f on a set D. Then f is continuous on D.

Theorem

Suppose that (f_n) is a sequence of continuous complex-valued functions converging uniformly to f on a set D. Then if Γ is any contour in D,

$$\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\,dz.$$

Theorem

Suppose that (f_n) is a sequence of analytic complex-valued functions converging uniformly to f on a domain D. Then f is analytic on D.

Power Series vs. Taylor Series

Theorem

Suppose that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a positive radius of convergence* $R>0$. *Then*

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{2}$$

is analytic in $D=B_R(z_0)$. *Moreover*

$$a_n=\frac{f^{(n)}(z_0)}{n!},$$

and (2) *is the Taylor series for* f *about* z_0.