# Residues

## Definition

If $f$ has an isolated singularity at $z_0$ with Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \tag{1}$$

for $z \in B'_R(z_0)$ with $R > 0$, then we call $b_1$ the residue of $f$ at $z_0$ and write $\operatorname{Res}(f; z_0) = b_1$.

## Remark

If $f$ has an isolated singularity at $z_0$ with Laurent series given by (1) in $B'_R(z_0)$ and if $\Gamma$ is any positively oriented contour in $B'_R(z_0)$ with $z_0$ in its interior, then

$$\int_\Gamma f(z)\, dz = 2\pi i \operatorname{Res}(f; z_0).$$

# Basics

**Proposition**

*If $f$ has a simple pole at $z_0$, then*

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

*Conversely, if $z_0$ is an isolated singularity for $f$ and*

$$\lim_{z \to z_0} (z - z_0) f(z) = L \neq 0,$$

*then $z_0$ is a simple pole and* $\mathrm{Res}(f; z_0) = L$.

# Simple Pole Lemma

**Lemma (Basic Simple Pole Lemma)**

*If $g$ and $h$ are analytic at $z_0$ such that $g(z_0) \neq 0$ and such that $h$ has a simple zero at $z_0$, then*

$$f(z) := \frac{g(z)}{h(z)}$$

*has a simple pole at $z_0$ and*

$$\mathrm{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

# General Poles

**Lemma**

*Suppose that $f$ has a pole of order $m$ at $z_0$. Then*

$$\mathrm{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right).$$

# Cauchy Residue Theorem

## Theorem (Cauchy Residue Theorem)

*Suppose that $f$ is analytic on and inside a simple closed contour $\Gamma$ except for isolated singularities at $z_1, \ldots, z_n$ inside of $\Gamma$. Then*

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}(f; z_k). \tag{2}$$

## Remark (Notation)

We often write (2) as

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{z \text{ inside } \Gamma} \operatorname{Res}(f; z)$$

with the understanding that the sum is finite since $\operatorname{Res}(f; z) = 0$ if $z$ is not a pole or essential singularity.