Riemann's Theorem

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1 A Proof of Riemann's Theorem

Theorem 1. *Let Γ be a contour in the plane and suppose that g is continuous on Γ. Let $D = \{\, z \in \mathbf{C} : z \notin \Gamma \,\}$. For each $n \geq 1$ define*

$$F_n(z) = \int_\Gamma \frac{g(w)}{(w-z)^n}\,dw \quad \text{for all } z \in D.$$

Then F_n is analytic on D and for all $z \in D$ we have

$$F_n'(z) = nF_{n+1}(z). \tag{1}$$

Proof. Fix $z_0 \in D$. Since D is open, there is a $\delta > 0$ such that $B_{2\delta}(z_0) \subset D$. Therefore

$$|w - z_0| \geq 2\delta \quad \text{for all } w \in \Gamma.$$

Since

$$\frac{1}{w-z} - \frac{1}{w-z_0} = \frac{z-z_0}{(w-z)(w-z_0)}, \tag{2}$$

it follows that

$$\begin{aligned} F_1(z) - F_1(z_0) &= \int_\Gamma \Big(\frac{1}{w-z} - \frac{1}{w-z_0}\Big) g(w)\,dw \\ &= (z-z_0)\int_\Gamma \frac{g(w)}{(w-z)(w-z_0)}\,dw \end{aligned} \tag{3}$$

Since Γ is a closed and bounded subset of $\mathbf{C}$, $M := \max_{w\in\Gamma} |g(w)|$ is finite. Thus if $0 < |z - z_0| < \delta$, then

$$|w - z| \geq |w - z_0| - |z - z_0| \geq \delta,$$

and it follows that

$$|F_1(z) - F_1(z_0)| \le \frac{M\ell(\Gamma)}{2\delta^2}|z - z_0|. \tag{4}$$

Using (4), it follows that

$$\lim_{z\to z_0} |F_1(z) - F_1(z_0)| = 0.$$

Therefore F_1 is continuous at z_0.

For each $n \ge 1$, let

$$G_n(z) = \int_\Gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,dw \quad \text{for all } z \in D.$$

Keep in mind that

$$G_n(z_0) = F_{n+1}(z_0). \tag{5}$$

Notice that

$$\tilde{g}(w) = \frac{g(w)}{w - z_0}$$

is continuous on Γ and that

$$G_n(z) = \int_\Gamma \frac{\tilde{g}(z)}{(w-z)^n}\,dz.$$

Thus we can repeat the argument above with $\tilde{g}$ in place of g to conclude that G_1 is continuous at z_0.

Part of the point of introducing G_1 is that using (3) we have

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\Gamma \frac{g(w)}{(w-z)(w-z_0)}\,dw = G_1(z).$$

Hence using (5) and the continuity of G_1 we have

$$F_2(z_0) = G_1(z_0) = \lim_{z\to z_0} G(z) = \lim_{z\to z_0} \frac{F_1(z) - F_0(z_0)}{z - z_0}.$$

That is, F_1 is differentiable at z_0 and $F_1'(z_0) = F_2(z_0)$.

Since $z_0 \in D$ was arbitrary, we've proved the result in the case that $n = 1$.

We now proceed by induction. Thus we assume that for some $n \ge 2$, we established that

$$F_{n-1}'(z_0) = (n-1)F_n(z_0).$$

It will suffice to show that $F_n'(z_0) = nF_{n+1}(z_0)$. Note that after replacing g by $\tilde{g}$, we can also assume that

$$G_{n-1}'(z_0) = (n-1)G_n(z_0).$$

After re-writing (2) as

$$\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z)(w-z_0)},$$

we have

$$\begin{aligned} F_n(z) &= \int_\Gamma \frac{g(w)}{(w-z)^n}\,dw = \int_\Gamma \frac{g(w)}{(w-z)^{n-1}(w-z)}\,dw \\ &= \int_\Gamma \frac{g(w)}{(w-z)^{n-1}(w-z_0)}\,dw + (z-z_0)\int_\Gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,dw \\ &= G_{n-1}(z) + (z-z_0)G_n(z). \end{aligned}$$

Plugging this into $F_n(z) - F_n(z_0) = F_n(z) - G_{n-1}(z_0)$ gives us

$$F_n(z) - F_n(z_0) = G_{n-1}(z) - G_{n-1}(z_0) + (z-z_0)G_n(z). \tag{6}$$

Assuming $0 < |z-z_0| < \delta$ as above, then

$$|G_n(z)| = \left|\int_\Gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,dw\right| \leq \frac{M\ell(\Gamma)}{2\delta^{n+1}}.$$

Therefore

$$|F_n(z) - F_n(z_0)| \leq |G_{n-1}(z) - G_{n-1}(z_0)| + \frac{M\ell\Gamma)}{2\delta^{n+1}}|z-z_0|. \tag{7}$$

Since G_{n-1} is differentiable at z_0, it must be continuous there. It then follows from (7) that

$$\lim_{z\to z_0} |F_n(z) - F_n(z_0)| = 0.$$

Therefore F_n is continuous at z_0 as is G_n.

We now use (6) to see that

$$\begin{aligned} \lim_{z\to z_0} \frac{F_n(z) - F_n(z_0)}{z-z_0} &= \lim_{z\to z_0}\Big(\frac{G_{n-1}(z) - G_{n-1}(z_0)}{z-z_0} + G_n(z)\Big) \\ &= G_{n-1}'(z_0) + G_n(z_0) \\ &= (n-1)G_n(z_0) + G_n(z_0) \\ &= nG_n(z_0) = nF_{n+1}(z_0). \end{aligned}$$

Thus $F'(z_0)$ exists and equals $nF_{n+1}(z_0)$ as required. Since z_0 is arbitrary, we're done. $\square$

Corollary 2. *Let g, Γ, and F_n be as in the statement of Theorem 1. Then F_1 has derivatives of all orders and*

$$F_1^{(n)}(z) = n!F_{n+1}(z) \quad \text{for all } z \in D.$$

Proof. We have $F_1'(z) = F_2(z)$ by Theorem 1 in the case that $n = 1$. Assume that for some $n \geq 2$, we've proved that $F_1^{(n-1)}(z) = (n-1)!F_n(z)$. Then by (1),

$$F_1^{(n)}(z) = (n-1)!F_n'(z) = n!F_{n+1}(z). \quad \square$$

Corollary 3. *Suppose that f is analytic on and inside a simple closed contour Γ and that D is the interior of Γ. Then f has derivatives of all orders on D and for each $z \in D$ and $n \geq 0$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Proof. Let F_n be as in Theorem 1 with $g = f$. Then the Cauchy Integral Formula implies that

$$F_1(z) = 2\pi i f(z) \quad \text{for all } z \in D.$$

Now by Corollary 2, f has derivatives of all orders and

$$2\pi i f^{(n)}(z) = F_1^{(n)}(z) = n!F_{n+1}(z) = n! \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, dw. \quad \square$$

Theorem 4. *Suppose that f is analytic in a domain D. Then f' is analytic in D. Therefore f has derivatives of all orders in D.*

Proof. Fix $z_0 \in D$. It will suffice to see that $f''(z_0)$ exists. Since D is open, there is a $\delta > 0$ such that $B_{2\delta}(z_0) \subset D$. Let C_δ be the positively oriented circle $|z - z_0| = \delta$. Then f is analytic on and inside of C_δ. Since z_0 lies inside of C_δ, it follows from Corollary 3 that $f''(z_0)$ exists. $\square$