Theorem (Maximum Modulus Principle)

Suppose that f is analytic on a domain D. If there is point $z_0 \in D$ such that

$$|f(z)| \leq |f(z_0)|$$
 for all $z \in D$,

⊡ ▶ ∢ ∃ ▶

then f is constant.

Remark (Bounded Regions)

- Recall that a domain D is bounded if there is a R > 0 such that D ⊂ B_R(0).
- The boundary ∂D of D is the set of points z such that every open ball $B_r(z)$ contains points in D and not in D.
- The closure \overline{D} of D is the union of D and ∂D . Of course, \overline{D} is closed.
- If D is bounded, then \overline{D} is closed and bounded.
- Thus if *D* is a bounded domain, then any continuous real-valued function on \overline{D} must attain its maximum and minimum on \overline{D} .

Theorem

Suppose that D is a bounded domain with closure $\overline{D} = D \cup \partial D$. Suppose also that $f : \overline{D} \subset \mathbf{C} \to \mathbf{C}$ is continuous and analytic on D. Then |f(z)| attains is maximum on ∂D .

Remark

In the text, the authors describe the hypotheses above by saying that "f is analytic on a bounded domain D and continuous up to and including its boundary". In class, we will use the formalism above.

Definition

A series of complex numbers is an expression of the form

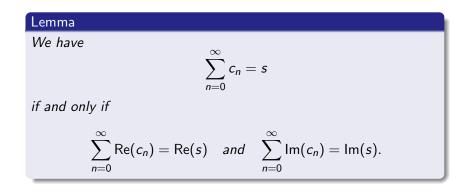
$$\sum_{n=0}^{\infty} c_n$$

(1)

with each $c_n \in \mathbf{C}$. The n^{th} partial sum of (1) is

$$s_n=c_0+c_1+\cdots+c_n.$$

We say that the series converges to $s \in \mathbf{C}$ if $\lim_{n\to\infty} s_n$ exists and equals s. Otherwise we say that the series diverges.



Theorem (Comparison Test)

Suppose that

$$|c_n| \leq a_n$$
 for all $n \geq N$.

Then if

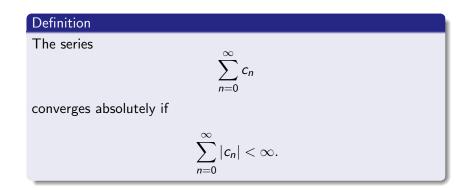
$$\sum_{n=0}^{\infty}a_n<\infty,$$

the complex series

$$\sum_{n=0}^{\infty} c_r$$

converges.

Absolute Convergence



Remark

Notice that by the comparison test, an absolutely convergent series is convergent.

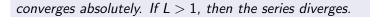
Theorem (Ratio Test)

Suppose that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=L.$$

 $\sum_{n=0}^{\infty} c_n$

If L < 1, then



Remark

Notice that having the limit exist is part of the hypotheses. In general, the limit might not exist. Also, if L = 1, the test gives no information.

Theorem (Geometric Series)

The geometric series

$$\sum_{n=0}^{\infty} ac^n = a + ac + ac^2 + \cdots$$

converges to

 $\frac{a}{1-c}$

イロト イヨト イヨト イヨト

if |c| < 1. Otherwise the series diverges.