Challenge Example From Lecture

Recall that the object was to describe the set

$$B = \{\, z = x + iy : |z - 1| + |z + 1| = 4 \,\}.$$

First some observations. Throughout, let $z = x + iy$. Then

$$z + \overline{z} = 2\,\mathrm{Re}(z) = 2x.$$

Also

$$|z + 1|^2 = (z + 1)(\overline{z} + 1) = |z|^2 + z + \overline{z} + 1 = |z|^2 + 2x + 1. \tag{1}$$

Similarly,

$$|z - 1|^2 = |z|^2 - 2x + 1. \tag{2}$$

Now if $z \in B$, then

$$|z + 1| = 4 - |z - 1|.$$

Therefore

$$|z + 1|^2 = 16 - 8|z - 1| + |z - 1|^2.$$

Now using (1) and (2), and canceling, we have

$$2x = 16 - 8|z - 1| - 2x \quad \text{or} \quad 4x = 16 - 8|z - 1| \quad \text{or} \quad x = 4 - 2|z - 1|.$$

Thus

$$2|z - 1| = 4 - x \quad \text{or} \quad 4|z - 1|^2 = 16 - 8x + x^2.$$

Using (2) again,

$$4(x^2 + y^2 - 2x + 1) = 16 - 8x + x^2.$$

Simplifying gives

$$3x^2 + 4y^2 = 12.$$

Now divide both sides by 12 to get that B is the locus of

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

as claimed in lecture.