1. (6) Complete the following ***definitions***.

 (a) A complex-valued function f on a set $D \subset \mathbf{C}$ is complex differentiable at $z_0 \in D$, if ...

 ANS: f is defined in a neighborhood of z_0 and the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

 exists.

 (b) If $f(z)$ is a set-valued function on a domain D, then a *branch* of $f(z)$ is ...

 ANS: a continuous function F on D such that $F(z) \in f(z)$ for all $z \in D$.

 (c) A real-valued function $u : D \subset \mathbf{R}^2 \to \mathbf{R}$ on a domain D is called *harmonic* if ...

 ANS: it has continuous second partials and $u_{xx} + u_{yy} = 0$ everywhere on D.

2. (6) Suppose that D is a domain and that $f : D \subset \mathbf{C} \to \mathbf{C}$ is given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

for real-valued functions u and v. Let $z_0 = a + ib$.

 (a) What are the Cauchy-Riemann equations for f at z_0?

 ANS: (a)

$$u_x(a, b) = v_y(a, b)$$
$$u_y(a, b) = -v_x(a, b).$$

 (b) How are the Cauchy-Riemann equations related to the complex differentiability of f at z_0? (Discuss both necessary and sufficient conditions.)

 ANS: If f is differentiable at z_0, then u and v have partial derivatives at z_0 and the Cauchy-Riemann equations must hold at z_0.

 The converse does not hold without additional assumptions. What we proved in class is that if the partial derivatives of u and v exist in a neighborhood of z_0, if they are continuous at z_0, and if the Cauchy-Riemann equations hold at z_0, then f is differentiable at z_0.

 Comment: Since I more or less told the class that this question would be on the exam, and since the exact question together with the solution was on *both* practice exams, I was fairly picky. You had to have separate necessary and sufficient conditions and you had to all three criteria for sufficiency clearly stated. Our first Cauchy Riemann theorem also told us that $f'(a+ib) = u_x(a, b) + iv_x(a, b) = v_y(a, b) - iu_y(a, b)$, but I didn't require this.

3. (6) Suppose that $f(z) = \dfrac{e^z}{z}$. Find formulas for real-valued functions u and v so that

$$f(x+iy) = u(x,y) + iv(x,y).$$

ANS: We have

$$\begin{aligned} f(x+iy) &= \frac{e^{x+iy}}{x+iy} = \frac{1}{x^2+y^2}\big((x-iy)e^x(\cos(y)+i\sin(y)\big) \\ &= \frac{e^x}{x^2+y^2}\big(x\cos(y)+y\sin(y)+i(x\sin(y)-y\cos(y)\big) \end{aligned}$$

Hence

$$u(x,y) = \frac{e^x(x\cos(y)+y\sin(y))}{x^2+y^2} \quad \text{and} \quad v(x,y) = \frac{e^x(x\sin(y)-y\cos(y))}{x^2+y^2}.$$

$u(x,y) =$ ______________________

$v(x,y) =$ ______________________

4. (8) Suppose that $p(z) = z^6 - 7z^3 - 8$.

(a) Find all the complex roots of $p(z)$. Your answers should be in the form $a+ib$ with $a, b \in \mathbf{R}$. (Hint: let $w = z^3$.)

(b) Factor $p(z)$ as a product of linear terms and irreducible quadratics all with real coefficients.

ANS: (a) We have $p(z) = (z^3+1)(z^3-8)$. Hence the roots are given by $z^3 = -1 = e^{i\pi}$ and $z^3 = 8 = e^{i0}$. The solutions to the first equation are $w_0 = e^{i\frac{\pi}{3}}$, $w_1 = -1$ and $w_2 = \overline{w_0}$. (Remember the non-real roots appear in conjugate pairs.) The solution to the second equation are $w_0 = 2$, $w_1 = 2e^{i\frac{2\pi}{3}}$, and $w_2 = \overline{w_1}$. Hence the roots are

$$\{\,\frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, \frac{1}{2} - i\frac{\sqrt{3}}{2}, 2, -1+i\sqrt{3}, -1-i\sqrt{3}\,\}$$

(b) Using the formula from lecture that $(z-w)(z-\bar{w}) = z^2 - 2\operatorname{Re}(w)z + |w|^2$:

$$p(z) = (z+1)(z-2)(z^2-z+1)(z^2+2z+4).$$

5. (6) Suppose that $f : D \subset \mathbf{C} \to \mathbf{C}$ is analytic on a domain D and that $f'(z) = 0$ for all $z \in D$. Prove that f is constant on D. (Yes, we proved this in lecture. Of course, you can't simply cite that result. As in the proof from lecture, you may use the fact that a real valued function on D whose first partials are everwhere zero is constant.)

ANS: Let $f(z) = u(z) + iv(z)$ as usual. Since $f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z)$, $f'(z) = 0$ implies that $u_x(z) = u_y(z) = 0$. Hence the partials of u vanish throughout D and u is constant. Similarly, the partials of v are zero everywhere and v is constant. This implies that f is constant.

6. (5) You may assume that

$$\frac{z^2+1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \quad \text{for constants } A,\ B, \text{ and } C.$$

What are A, B, and C?

ANS:

$$A = \lim_{z\to 1} \frac{z^2+1}{(z-2)^2} = 2,$$
$$B = \lim_{z\to 2} \frac{d}{dz}\Big(\frac{z^2+1}{z-1}\Big) = \lim_{z\to 2} \frac{2z(z-1)-(z^2+1)}{(z-1)^2} = -1 \quad \text{and}$$
$$C = \lim_{z\to 2} \frac{z^2+1}{z-1} = 5.$$

$A =$ __________ $B =$ __________ $C =$ __________

Math 43 — Exam I — Take Home Portion

Problems #1–#6 are to be completed in class on Friday. The remaining problems are to be turned in at the beginning of class on Monday (April 22^{nd}). Your solutions to the take-home portion are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. The in-class portion of the exam is "closed book". For the take-home portion, you may consult the text and your class notes, but no other sources are allowed. In both cases, you are to work alone and neither give nor receive help from anyone excepting only that you may ask me for clarification.

7. (6) Recall that the hyperbolic cosine and sine are given, respectively, by

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Let $u(x,y) = \cosh(x)\cos(y) + xy$.

(a) Find a function v such that $f(x+iy) = u(x,y) + iv(x,y)$ is entire.

(b) Find and expression for the derivative, $f'(x+iy)$, in terms of u and v.

ANS: (a) If v exists, then the Cauchy-Riemann equations must hold and

$$v_x(x,y) = -u_y(x,y) = \cosh(x)\sin(y) - x.$$

Therefore,

$$v(x,y) = \sinh(x)\sin(y) - \frac{x^2}{2} + C(y).$$

But then

$$v_y(x,y) = \sinh(x)\cos(y) + C'(y) = u_x(x,y) = \sinh(x)\cos(y) + y.$$

Therefore we need $C'(y) = y$ and $C(y) = \frac{y^2}{2} + c$. Therefore, we want

$$v(x,y) = \sinh(x)\sin(y) + \frac{y^2 - x^2}{2} + c.$$

Then $f = u + iv$ satisfies the Cauchy-Riemann equations everywhere and both u and v have continuous partials everywhere. Hence by our second Cauchy-Riemann Theorem, f is entire.

(b) By our first Cauchy-Riemann Theorem,

$$f'(x+iy) = u_x(x,y) + iv_x(x,y) = \sinh(x)\cos(y) + y + i(\cosh(x)\sin(y) - x).$$

8. (6) Suppose that f is complex differentiable at z_0. Show that f must be continuous at z_0. (Hint: f is continuous at z_0 if and only if $\lim_{z\to z_0}\big(f(z) - f(z_0)\big) = 0$.)

ANS: We have

$$\lim_{z\to z_0}\big(f(z) - f(z_0)\big) = \lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0}(z - z_0) = f'(z_0)\cdot 0 = 0.$$

9. (6) Let $z = x + iy$. Given conditions on x and y so that $\operatorname{Log}(e^{z^2}) = z^2$.

ANS: We have $z^2 = x^2 - y^2 + i2xy$. Therefore $e^{z^2} = e^{x^2-y^2}e^{i2xy}$. It follows that

$$\log(e^{z^2}) = \ln\big(e^{x^2-y^2}\big) + i\arg\big(e^{i2xy}\big) = x^2 - y^2 + i\{\,2xy + 2\pi k : k \in \mathbf{Z}\,\}.$$

Therefore

$$\operatorname{Log}(e^{z^2}) = x^2 - y^2 + i2xy$$

exactly when $2xy \in (-\pi, \pi]$. That is, when $xy \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big]$.

10. (6) Suppose that v is a harmonic conjugate for u in a domain D. Show that $u^3 - 3uv^2$ is harmonic on D. You may assume u and v have continuous second partials.

ANS: By definition, $f(z) = u(z) + iv(z)$ is analytic on D. But then so is $h(z) = f(z)^3 = u(z)^3 - 3u(z)v(z)^2 + i(-3u(z)^2v(z) - v(z)^3)$. Since $u^3 - 3uv^2$ is the real part of h, it must be harmonic as we proved in lecture.

COMMENT: I never dreamed some folks would try to do this by the tedious calculation of the second partials—we're supposed to be beyond the chain rule and calculating partial derivatives. If you did use that method, I demanded an coherent argument where the steps were obvious and *clearly* justified. Writing down some massive expression and crossing out various bits without explanation did not cut it.

11. (8)(a) Suppose that f and g are analytic branches of $z^{\frac{1}{5}}$ on a domain D such that $0 \notin D$. Show that there is a fifth root, w_0, of 1 such that $f(z) = w_0 g(z)$ for all $z \in D$. I suggest considering $h(z) = f(z)/g(z)$.

(b) Now suppose that $D = D^* = \mathbf{C} \setminus (-\infty, 0]$. Let f be an analytic branch of $z^{\frac{1}{5}}$ in D^* such that $f(1) = 1$. Show that $f(z) = \exp\bigl(\frac{1}{5}\operatorname{Log}(z)\bigr)$ for all $z \in D^*$.

ANS: (a) Let $h(z) = f(z)/g(z)$. Since neither f nor g vanishes on D, h is analytic and non-vanishing on D. Since $h(z)^5 = 1$, the chain rule implies $5h(z)^4h'(z) = 0$ for all $z \in D$. Since h is never zero, we have $h'(z) = 0$ for all $z \in D$ and h is constant. But if $h(z) = w_0$ for all z, then $w_0^5 = 1$. Therefore $f(z) = w_0 g(z)$ for all $z \in D$ as required.

(b) Clearly, $g(z) = \exp\bigl(\frac{1}{5}\operatorname{Log}(z)\bigr)$ is a branch of $z^{\frac{1}{5}}$ in D^*. By part (a), there is a fifth root of unity w_0 such that

$$f(z) = w_0 \exp\Bigl(\frac{1}{5}\operatorname{Log}(z)\Bigr) \quad \text{for all } z \in D^*.$$

Since $\operatorname{Log}(1) = 0$, $\exp\bigl(\frac{1}{5}\operatorname{Log}(1)\bigr) = 1$ and $w_0 = 1$.

12. (6) Suppose that D is a domain and that $0 \notin D$. Suppose also that g is analytic on D and that $g'(z) = \frac{1}{z}$ for all $z \in D$. Show that there is a $w_0 \in \mathbf{C}$ such that

$$\exp(g(z) + w_0) = z \quad \text{for all } z \in D.$$

I suggest you consider $h(z) = \exp(g(z))/z$.

ANS: Let h be as suggested. Note that h is never zero. Then for all $z \in D$, we have

$$h'(z) = \frac{g'(z)e^{g(z)}z - e^{g(z)}}{z^2} = \frac{\frac{1}{z}e^{g(z)}z - e^{g(z)}}{z^2} = 0.$$

Hence there is a nonzero constant c so that

$$e^{g(z)} = cz \quad \text{for all } z \in D.$$

Let $w_0 \in \log\bigl(\frac{1}{c}\bigr)$ so that $e^{w_0} = \frac{1}{c}$. Then

$$e^{g(z)+w_0} = cz\frac{1}{c} = z$$

as required.
COMMENT: You've just shown that there is an analytic branch of $\log z$ in any domain where $\frac{1}{z}$ has an antiderivative.

NAME : ________________________________

Math 43

19 April 2019

Dana P. Williams

Problems #1–#6 are to be completed in class on Friday. The remaining problems are to be turned in at the beginning of class on Monday (April 22nd). Your solutions to the take-home portion are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. The in-class portion of the exam is "closed book". For the take-home portion, you may consult the text and your class notes, but no other sources are allowed. In both cases, you are to work alone and neither give nor receive help from anyone excepting only that you may ask me for clarification.

FERPA RELEASE: Because of privacy concerns, I am not allowed to return your graded exams to the homework boxes without your permission. If you wish me to return your exam via your homework box, please sign on the line indicated below. Otherwise, you will have to pick your exam up in my office after the exams have been returned in lecture.

SIGN HERE: ____.

Problem	Points	Score
1	6	
2	6	
3	6	
4	8	
5	6	
6	5	
7	6	
8	6	
9	6	
10	6	
11	8	
12	6	
Total	75	