Selected Solutions for m43s20 Homework 10 (Last)

Last Updated: June 1, 2020

#2, §6.7 Buy the Fundamental Theorem of Algebra, every polynomial with complex coefficients factors completely into linear factors. Hence every polynomial P of degree n has n zeros counted up to multiplicity. Since there are only finitely many zeros, there is a R such that all the zeros of P lie inside the circle $|z| = R$. Hence by our favorite homework problem (EP-2),

$$N_P = n = \frac{1}{2\pi i} \int_{|z|=R} \frac{P'(z)}{P(z)}\, dz$$

But this is (essentially) what we were asked to prove.

#4, §6.7 Let $g(z) = f(z) - w_0$. By assumption g does not vanish on the circle $|z| = \rho$. Hence the number of zeros of g inside $|z| = \rho$ counted up to multiplicity is

$$N_g = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{g'(z)}{g(z)}\, dz = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f'(z)}{f(z) - w_0}\, dz.$$

By the zeros of g are the solutions to $f(z) - w_0$. Of course, we are counting up to multiplicity here.

#6, §6.7 Let $f(z) = 4z^2$ and $g(z) = z^6 + 4z^2 - 1$. Then on $|z| = 1$,

$$|f(z) - g(z)| = |1 - z^6| \leq 2 < 4 = |4z^2|.$$

Hence f and g have the same number of zeros inside $|z| = 1$ by Rouche's Theorem. Since f has two zeros up to multiplicity inside $|z| = 1$, so does g.