Math 43: Spring 2020
Lecture 22 Part II

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Characterizing Poles

Corollary (Before the Break)

If f has a pole of order $m \geq 1$ at z_0, then

$$\lim_{z \to z_0} \left| (z - z_0)^k f(z) \right| = \infty$$

whenever $0 \leq k < m$.

Theorem

Suppose that f has an isolated singularity at z_0. Then f has a pole at z_0 if and only if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Proof.

If f has a pole at z_0, then $\lim_{z \to z_0} |f(z)| = \infty$ by the Corollary with $k = 0$.

The Converse

Proof.

Now suppose that $\lim_{z \to z_0} |f(z)| = \infty$. Let

$$g(z) = \frac{1}{f(z)}.$$

Then g has an isolated singularity at z_0. Also, $\lim_{z \to z_0} g(z) = 0$.
Hence z_0 is a removable singularity for g and

$$h(z) = \begin{cases} g(z) & \text{if } z \neq z_0, \text{ and} \\ 0 & \text{if } z = z_0. \end{cases}$$

is necessarily analytic at z_0! Moreover, since $g(z)$ is not zero near z_0, h is not constant. Hence z_0 has finite order $m \geq 1$.

Proof.

Since h has a zero of order $m \geq 1$ at z_0, we have

$$h(z) = (z - z_0)^m p(z)$$

where p is analytic at z_0 and $p(z_0) \neq 0$. But now if $z \neq z_0$

$$f(z) = \frac{1}{h(z)} = \frac{1}{(z - z_0)^m} \cdot \frac{1}{p(z)} = \frac{r(z)}{(z - z_0)^m},$$

where r is analytic at z_0 and $r(z_0) = \frac{1}{p(z_0)} \neq 0$. Hence f has a pole of order m at z_0 in view of our previous result. $\qquad\square$

Essential Singularities

Corollary

Suppose that f has an isolated singularity at z_0. Then f has an essential singularity at z_0 if and only if $|f(z)|$ is not bounded near z_0 and $\lim_{z \to z_0} |f(z)| \neq \infty$.

Proof.

By default. This is the only case left. We know that z_0 is removable if and only if $|f(z)|$ is bounded near z_0, and we know that z_0 is a pole if and only if $\lim_{z \to z_0} |f(z)| = \infty$. $\qquad\square$

Example

Consider $f(z) = e^{\frac{1}{z}} = \exp\left(\frac{1}{z}\right)$.

Let $x \in \mathbf{R}$.

Then $\left|f\left(\frac{1}{ix}\right)\right| = 1$ for all $x \neq 0$. Therefore $\lim_{z \to 0} |f(z)| \neq \infty$.

On the other hand for real x, $\lim_{x \searrow 0} |e^{\frac{1}{x}}| = \infty$ and $|f(z)|$ is not bounded near 0.

This means that 0 is an essential singularity for f. Of course, we knew this from the start because we can write down the Laurent series for f: $f(z) = 1 + \sum_{j=1}^{\infty} \frac{1}{j! \cdot z^j}$.

Definition

A subset $D \subset \mathbf{C}$ is dense in $\mathbf{C}$ if for all $w \in \mathbf{C}$ and all $\epsilon > 0$ we have $D \cap B_\epsilon(w) \neq \emptyset$.

Remark

D is dense in $\mathbf{C}$ if and only if given $w \in \mathbf{C}$ there is a sequence $(d_n) \subset D$ such that $d_n \to w$.

Example

1. $R = \{\, r + is : r, s \in \mathbf{Q} \,\}$ is dense in $\mathbf{C}$.

2. $\mathbf{C} \setminus R$ is also dense in $\mathbf{C}$.

3. If $F \subset \mathbf{C}$ is a finite set, then $\mathbf{C} \setminus F$ is dense in $\mathbf{C}$.

Lemma

If f is a non-constant entire function, then f has dense range. That is, $f(\mathbf{C})$ is dense in $\mathbf{C}$.

Proved on the Midterm.

You proved this on the exam. If $f(\mathbf{C})$ is not dense, then there is a $w \in \mathbf{C}$ and an $\epsilon > 0$ such that $f(\mathbf{C}) \cap B_\epsilon(w) = \emptyset$. In other words, $|f(z) - w| \geq \epsilon$ for all z. Then $g(z) = \frac{1}{f(z)-w}$ is entire and $|g(z)| \leq \frac{1}{\epsilon}$. Then g is constant. If $g(z) = c$ for all z, then $c \neq 0$ and $f(z) = \frac{1}{c} + w$ is constant. $\qquad\square$

Theorem (Casorati-Weierstrass)

Suppose that f has an essential singularity at z_0. Then for all $\epsilon > 0$, the image, $f\left(B'_\epsilon(z_0)\right)$, of the deleted ϵ-ball about z_0 is dense in $\mathbf{C}$.
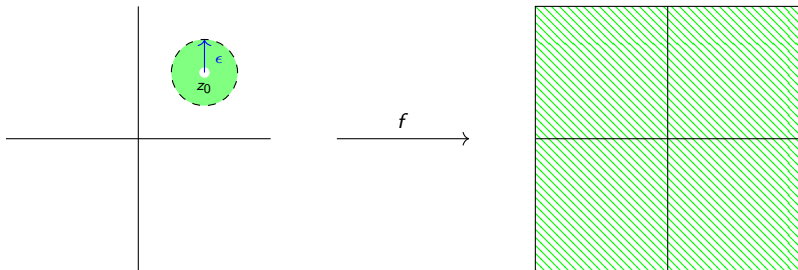
The Proof

Proof.

Suppose to the contrary that there is a $\epsilon > 0$ such that $f\left(B'_\epsilon(z_0)\right)$ is not dense. Then there is a $w \in \mathbf{C}$ and a $r > 0$ such that

$$|f(z) - w| \geq r \quad \text{for all } z \in B'_\epsilon(z_0).$$

Let $g(z) = \dfrac{1}{f(z) - w}$. Then g has an isolated singularity at z_0 and

$$|g(z)| \leq \frac{1}{r} \quad \text{for all } z \in B'_\epsilon(z_0).$$

Hence g has a removable singularity at z_0!

One More Case

Proof.

Since z_0 is a removable singularity for g, there is a $L \in \mathbf{C}$ such that $\lim_{z \to z_0} g(z) = L$. If $L \neq 0$, then

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{1}{g(z)} + w = \frac{1}{L} + w.$$

But then f would have a removable singularity at z_0 which contradicts our hypothesis. But if $L = 0$, then

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| \frac{1}{g(z)} + w \right| = \infty.$$

Then f would have a pole at z_0. This is also contradicts our hypothesis. Hence $f\big(B'_\epsilon(z_0)\big)$ must be dense as claimed. $\qquad\square$

I am stating these for the sake of intellectual curiosity. They are not fair game in this course.

Theorem (Picard's Great Theorem)

If f has an essential singularity at z_0, then with possibly one exception, f assume every complex value infinitely often in any deleted neighborhood $B'_\epsilon(z_0)$ of z_0.

Remark

You can verify this for $f(z) = \exp(\frac{1}{z})$ near $z_0 = 0$. Note that the exceptional value here is 0.

Theorem (Picard's Little Theorem)

If the range of an entire function omits more than one point, then it is constant.

Break Time

Remark

There is a lot in §5.6. Note that we have covered some material not in the text!

That is definitely enough for today!!