Math 43: Spring 2020
Lecture 24 Part I

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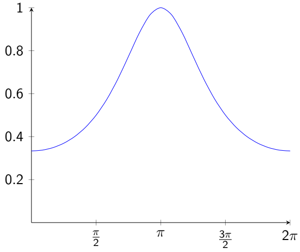
Time for a Calculus Problem in the "Real" World

Example

Compute $I = \int_0^{2\pi} \frac{1}{2+\cos(\theta)}\,d\theta$.



Remark

While it is possible to compute this integral using classical techniques—I will give a general method for the fun of it on the assignments page—it is quite involved. The game here is to use residues to find the solution quite a bit more easily.

The Game

First, observe that, if as usual we assume $|z| = 1$ is positively oriented, then we can use the parameterization $z(\theta) = e^{i\theta}$ with $\theta \in [0, 2\pi]$ to evaluate as follows:

$$\int_{|z|=1} F(z)\,dz = \int_0^{2\pi} F(e^{i\theta}) ie^{i\theta}\,d\theta.$$

Furthermore, if $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$ and

$$\cos(\theta) = \frac{1}{2}\Big(z + \frac{1}{z}\Big) \quad \text{and} \quad \sin(\theta) = \frac{1}{2i}\Big(z - \frac{1}{z}\Big).$$

The point is that if we let

$$F(z) = \frac{1}{2 + \frac{1}{2}\big(z + \frac{1}{z}\big)} \cdot \frac{1}{iz},$$

then

$$I = \int_{|z|=1} F(z)\,dz.$$

Then we use residues to compute that

$$I = \int_{|z|=1} F(z)\,dz = \int_{|z|=1} \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{1}{iz}\frac{2}{2}\,dz$$
$$= \int_{|z|=1} \frac{-2i}{4z+z^2+1}\,dz = \int_{|z|=1} \frac{-2i}{z^2+4z+1}\,dz$$

which, using the quadratic formula, is

$$= \int_{|z|=1} \frac{-2i}{(z-p)(z-q)}\,dz$$

where $p = -2+\sqrt{3}$ and $q = -2-\sqrt{3}$. Since only p lies inside $|z| = 1$,

$$I = 2\pi i \cdot \operatorname{Res}(p) = 2\pi i \cdot \lim_{z\to p} \frac{-2i}{z-q} = 2\pi i \cdot \frac{-2i}{p-q}$$
$$= 2\pi i \cdot \frac{-2i}{2\sqrt{3}} = \boxed{\frac{2\pi}{\sqrt{3}}}.$$

General Method

In general, consider a rational function $U(x,y)$ defined on $[-1,1]\times[-1,1]$. For example,

$$U(x,y)=\frac{1}{2+x} \quad \text{or} \quad U(x,y)=\frac{xy}{1+x^2+2y^2}.$$

Then we have

$$\int_0^{2\pi} U\big(\cos(\theta),\sin(\theta)\big)\,d\theta=\int_{|z|=1} F(z)\,dz,$$

where

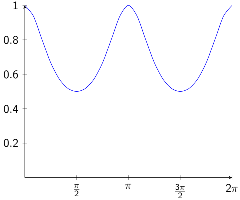
$$F(z)=U\Big[\frac{1}{2}\big(z+\frac{1}{z}\big),\frac{1}{2i}\big(z-\frac{1}{z}\big)\Big]\cdot\frac{1}{iz}.$$

Note that $U(x,y)=\frac{1}{2+x}$ gives us the example we just worked out!

Another Example

Example

Calculate $I = \displaystyle\int_0^{2\pi} \frac{1}{1+\sin^2(\theta)}\,d\theta.$



Here

$$I = \int_{|z|=1} F(z)\,dz,$$

where

$$F(z) = \frac{1}{1+\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^2}\cdot\frac{1}{iz} = \frac{1}{iz}\cdot\frac{1}{1-\frac{1}{4}\left(z^2-2+\frac{1}{z^2}\right)}\cdot\frac{4z^2}{4z^2}$$

$$= \frac{-4iz}{z^2\left(4-z^2+2-\frac{1}{z^2}\right)} = \frac{4iz}{z^4-6z^2+1}$$

Continued

Thus, viewing the denominator as a quadratic in z^2, we have

$$F(z) = \frac{4iz}{z^4 - 6z^2 + 1} = \frac{4iz}{(z^2 - p)(z^2 - q)}$$

with $p = 3 - 2\sqrt{2}$ and $q = 3 + 2\sqrt{2}$. Note that $pq = 1$, and hence $|p| \cdot |q| = 1$. Therefore the only singularities inside $|z| = 1$ are $\pm\sqrt{p}$ and

$$F(z) = \frac{4zi}{(z - \sqrt{p})(z + \sqrt{p})(z^2 - q)}$$

Therefore by Cauchy's Residue Theorem

$$I = 2\pi i \big[\operatorname{Res}(\sqrt{p}) + \operatorname{Res}(-\sqrt{p})\big].$$

Computing the Residues

We can compute $\operatorname{Res}(\pm\sqrt{p})$ by brute force, or we can use the Simple Pole Lemma applied to

$$F(z) = \frac{g(z)}{h(z)} = \frac{4zi}{z^4 - 6z^2 + 1}.$$

Hence

$$\operatorname{Res}(\pm\sqrt{p}) = \frac{4zi}{4z^3 - 12z}\Big|_{z=\pm\sqrt{p}} = \frac{i}{z^2 - 3}\Big|_{z=\pm\sqrt{p}} = \frac{i}{p-3} = -\frac{i}{2\sqrt{2}}.$$

Hence

$$I = 2\pi i \cdot 2\operatorname{Res}(\sqrt{p}) = 2\pi i \cdot \frac{-i}{\sqrt{2}} = \pi\sqrt{2}.$$

Reality Check!

Remark (Real Answers)

One of the cool things about this process is that we use complex theory to provide answers to questions about real calculus! In particular, our answers must be real numbers! This provides a handy reality check (pun intended!) to guard against algebraic mistakes.

Authors Showing Off

In Example 2 in §6.2, the authors try to scare us by asking for something like

$$I = \int_0^{\pi} \frac{1}{2+\cos(\theta)}\,d\theta.$$

Since the integral is taken over $[0,\pi]$ rather than $[0,2\pi]$, we seem to be stuck. But we can take advantage of symmetry there. If we make the change of variables $t = 2\pi - \theta$, then since $\cos(2\pi - t) = \cos(t)$, we have

$$\int_{\pi}^{2\pi} f(\cos(\theta))\,d\theta = \int_{\pi}^{0} f(\cos(2\pi - t))(-1)\,dt = \int_0^{\pi} f(\cos(t))\,dt.$$

Hence $\int_0^{2\pi} f(\cos(\theta))\,d\theta = 2\int_0^{\pi} f(\cos(\theta))\,d\theta$. Therefore

$$I = \frac{1}{2}\int_0^{2\pi} \frac{1}{2+\cos(\theta)}\,d\theta = \frac{1}{2}\Bigl(\frac{2\pi}{\sqrt{3}}\Bigr) = \frac{\pi}{\sqrt{3}}.$$

Break Time

Remark

Is the same true if we replace $\cos(\theta)$ by $\sin(\theta)$? Not quite. Sounds like a good homework problem to me.

Time for a Break.