Math 43: Spring 2020
Lecture 25 Part I

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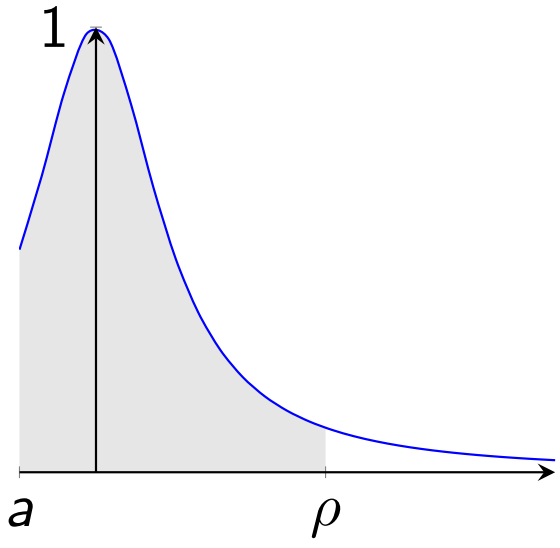
Wednesday May 27, 2020

Improper Integrals



Let's remind ourselves about about improper Riemann Integrals from back in the day. If f is continuous on $[a, \infty)$, then we define

$$\int_a^\infty f(x)\,dx = \lim_{\rho\to\infty} \int_a^\rho f(x)\,dx$$

provided the limit exists. If the limit exists, we say that the integral converges. Otherwise, we say the integral diverges. The idea is, at least in the case that in the case $f(x) \geq 0$ for all x, the limiting value represents the area of the infinite region under the curve above $[a, \infty)$.

Two Sided Improper Integrals

Naturally, we also define

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{\rho\to\infty} \int_{-\rho}^{a} f(x)\,dx.$$

Our contour integral methods for evaluation improper integrals typically involve integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx$$

for a continuous function f on $(-\infty,\infty)$. These are defined to converge only if for some, and hence any, $a \in (-\infty,\infty)$, both

$$\int_{-\infty}^{a} f(x)\,dx \quad \text{and} \quad \int_{a}^{\infty} f(x)\,dx$$

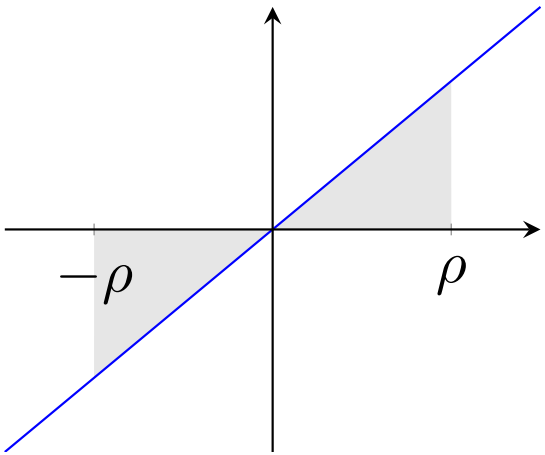
converge.

Pedantry

The reason for this apparent pedantry is to make sure we are measuring something meaningful. We do not want to say that just because

$$\int_{-\infty}^{\infty} x\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} x\,dx = 0, \qquad (\dagger)$$

that then the integral converges.



The idea is that $\int_{-\infty}^{\infty} f(x)\,dx$ should represent the area trapped above the x-axis minus the area trapped below the x-axis. This doesn't really make sense if both these areas are infinite and we just get lucky that the areas cancel due to symmetry in $(\dagger)$

Principal Value Integrals

However, symmetric limits, such as (†) on the previous slide are easier to evaluate in practice. Moreover, they are what our contour integral techniques will evaluate. So, just as the authors of our text do, we make a definition, and call

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx$$

the principal value of $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.

Some Good Stuff

Here are some handy calculus results that we will use and not prove. Here f and g are always meant to be a continuous functions on $(-\infty, \infty)$.

Lemma

If $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ *converges, then* $\displaystyle\text{p.v.}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$

Of course, we know from the example of $f(x) = x$, that the principal value can exist without the integral converging.

Lemma (Comparison Theorem)

If $|g(x)| \le f(x)$ *for all* x *and* $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx < \infty$, *then* $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx$ *converges.*

A Family of Tame Examples

Theorem (Plus Two for Converence)

Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients such that $\deg p(x) + 2 \leq \deg q(x)$. Then if $q(x)$ has no real roots,

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$$

converges.

Example

$\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2}\,dx$ converges. Hence

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2}\,dx = \text{p. v.}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2}\,dx.$$

Break Time

Remark

If you're wondering what happened to complex analysis, don't worry. We'll get back to that in the second part of the lecture. When we do, we'll know a little more about improper integrals than expected in the text. This will allow us to simplify a few things. But this means some of our results will differ from those in the text.

Time for a Break.