Math 43: Spring 2020
Lecture 26 Part II

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Recall

Theorem (Plus One Residue Theorem)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 1 \le \deg q(z)$ and such that $q(z)$ has no real roots. Let $a > 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Bigl[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Bigr] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Bigl[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Bigr]$$

An Example

Example

Compute $I = \displaystyle\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+2x+1}\,dx$ where $a > 0$.

Solution.

Well, there really isn't much to do here except to apply the "Plus One Residue Theorem" properly. We need $F(z) = \dfrac{ze^{iaz}}{z^2+2z+2}$. Then

$$I = \operatorname{Im}\big(2\pi i \operatorname{Res}(F; -1+i)\big).$$

Continued

Solution Continued.

By the Simple Pole Lemma

$$\begin{aligned}\operatorname{Res}(F;-1+i) &= \frac{ze^{iaz}}{2z+2}\bigg|_{z=-1+i} = \frac{(-1+i)e^{-ia-a}}{-2+2i+2}\\ &= \frac{e^{-a}}{2i}(-1+i)(\cos(a)-i\sin(a))\\ &= \frac{e^{-a}}{2i}\big[\sin(a)-\cos(a)+i\big(\cos(a)+\sin(a)\big)\big]\end{aligned}$$

Thus

$$I = \operatorname{Im}(2\pi i \operatorname{Res}(F;-1+i)) = \frac{\pi}{e^a}\big(\cos(a)+\sin(a)\big). \quad \square$$

Break Time

Remark

After a Mathematica Demo, we'll see that we can't always use the contour Γ_R as we did the the Plus Two and Plus One Residue Theorems.

There will be a break after the Demo.